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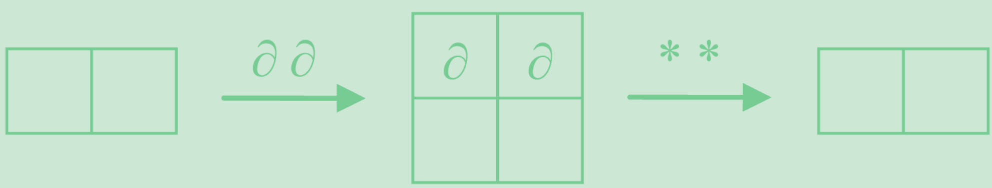

Extensions of Three-Dimensional Higher-Derivative Gravity

Yihao Yin

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Chapter 1

Introduction

1.1 General Relativity

Around 1915, based on his earlier work of special relativity and the principle that laws of physics should be the same in all reference frames including non-inertial ones, Einstein developed the theory of general relativity, in which gravity is treated as the curvature of spacetime, and the dynamics of the spacetime metric $g_{\mu\nu}$ is described by the Einstein's field equations[1]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \ , \tag{1.1.1}$$

where the R's contain second-order derivatives of the metric and the stress-energy tensor $T_{\mu\nu}$ represents the matter source. G is the Newton's constant and c is the speed of light. Furthermore, the source-free Einstein's equations can be integrated into the following action

$$S_{\text{EH}} = \int d^4x \sqrt{-g} R \ , \tag{1.1.2}$$

well-known as the Einstein-Hilbert action, where g is the determinant of the metric.

General relativity is a very successful theory in the sense that it can provide very precise predictions of many astronomical observations.

For instance, long before general relativity was established, astronomers discovered that the perihelion of the Mercury orbit is precessing at a rate that cannot be explained by the Newtonian gravity, even if the perturbative effects from other planets have been taken into account. On this issue, general relativity gives corrections to the Newtonian gravity that are not negligible in the gravitational field near the sun, and it indeed precisely fits all the observational data so far. Another famous example is the light deflection around a heavy object. According to general relativity, light will bend, when it passes by the sun, at a different angle from the prediction of the Newtonian gravity. Arthur Eddington led the first observational test in 1919 by measuring the star-image shift around the sun during a solar

[1] See Appendix A for the use of notations and conventions, e.g. the definition of the Ricci tensor / scalar.

eclipse, and a number of observations with improved precisions were performed in the following decades. The results of these observations all agree with general relativity.

Despite the huge success, general relativity however is still imperfect and facing challenges.

For instance, at very large distances. i.e. when general relativity is applied to cosmology, it is directly challenged by observational facts. Considering the attractive effect coming from the matter, we would expect the expansion of the universe is decelerating, but in 1998 it was discovered that the expansion is actually accelerating [1, 2], which means the universe must be filled with some kind of "dark energy", which acts repulsively. The dark energy is dominating our current universe, but its physical nature remains unknown.

On the other hand, we also do not understand how gravity behaves at very short distances. We expect that quantum effects of gravity are significant at the Planck scale (10^{-35}m), which is important for understanding the early universe soon after the Big Bang. However, it is difficult to quantize general relativity in the way we quantize particle physics, because doing this gives infinities that physicists cannot handle.

In view of these issues, physicists have been trying many different ways to modify general relativity. One type of modified general relativity is called "massive gravity", which will be discussed below.

1.2 Massive Gravity

One way of modifying general relativity is to make the graviton massive. From the point of view of particle physics, general relativity can be seen as the theory which describes massless spin-2 particles that mediate the gravitational force, i.e. it is the theory of massless gravitons. On the other hand, we can also construct models with massive gravitons as modifications to general relativity. We name such models the *massive gravity* [2].

One of the motivations of massive gravity is related to the dark energy. General relativity can accommodate the dark energy, provided that one more term is added to the Einstein's field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4} T_{\mu\nu} \ , \tag{1.2.1}$$

where the Λ in the extra term is called the cosmological constant, and it must have a positive value to make the dark energy repulsive. According to observational

[2]For a comprehensive review of massive gravity theories, see [3].

data, the value of Λ is non-vanishing but extremely small:

$$\Lambda \sim 10^{-52}\mathrm{m}^{-2} \sim 10^{-122}\ell_{\mathrm{P}}^{-2} \ ,$$

where ℓ_{P} is the Planck length. Many physicists believe that, in the system of Planck units ($\ell_{\mathrm{P}} = 1$) where "god-given" parameters like G and c are scaled to unity, Λ should also be of order unity, if it is a fundamental constant of nature. In this sense, the value 10^{-122} is "unnaturally" small. According to some literature (e.g. [4, 5, 6, 7]), giving mass to the graviton may solve this unnaturalness problem. Intuitively, one can think of the Yukawa potential: if we put a matter source at the origin of the coordinates, the potential V produced at a distance r in a massive gravity theory with the graviton mass m should be roughly

$$V \sim -\frac{e^{-mr}}{r} \ .$$

At a short distance $r \ll \frac{1}{m}$, this potential is effectively the same as the one in the massless gravity theory $V \sim -\frac{1}{r}$. However, at a large distance $r \gtrsim \frac{1}{m}$, i.e. when talking about the cosmic expansion, the mass m weakens the effect of gravity. In this situation, perhaps the true cosmological constant may be of order unity, while its effect in the cosmic expansion is weakened, which leads to a seemingly small value of Λ in (1.2.1) as an effective description.

Besides the phenomenological purposes, there are also other motivations of massive gravity from the purely theoretical perspective. For decades, various massive gravity models have been constructed, with all kinds of good or bad properties. The aims are not only to better fit experimental data, but also to better understand our currently standard theories. For instance, by studying massive gravity, we may know better in which sense and to what extent the theory of general relativity is unique, and whether it can be embedded in a larger framework of models. In this thesis, I will mainly focus on such purely theoretical discussions.

Let us start our discussion from the simplest and perhaps the earliest investigated model: the Fierz-Pauli theory for spin-2 [8], which can be seen as the linearized approximation of a massive gravity theory around the flat background. Its action reads

$$S = \int d^4x \left\{ h^{\mu\nu} G^{\mathrm{lin}}_{\mu\nu}(h) - m^2 \left(h^{\mu\nu}h_{\mu\nu} - h^2 \right) \right\} \ , \tag{1.2.2}$$

which is merely the linearized Einstein-Hilbert action with additional mass terms, where the symmetric tensor $h_{\mu\nu}$ is the graviton field, $h \equiv \eta^{\mu\nu}h_{\mu\nu}$ and

$$G^{\mathrm{lin}}_{\mu\nu}(h) = \Box h_{\mu\nu} - 2\partial_{(\mu}\partial^{\rho}h_{\nu)\rho} + 2\partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h \ , \tag{1.2.3}$$

which is the linearized Einstein tensor.

Naively one would expect the massless theory should be exactly the same as the $m \to 0$ limit of the massive theory. However, in 1970 Van Dam, Veltman and Zakharov discovered that this is not true [9, 10]. The key idea is that the 4-dimensional

Einstein-Hilbert action represents 2 physical degrees of freedom (corresponding to the two polarizations of the gravitational wave), but the massive theory (1.2.2) contains 5 degrees of freedom, which makes a difference that does not vanish by taking the limit. The massive graviton in the zero-mass limit can be seen as three massless particles which are spin-2, spin-1 and spin-0, with 2, 2 and 1 degrees of freedom, respectively. If we put a matter source into the theory, i.e. add the term $h^{\mu\nu}T_{\mu\nu}$ to (1.2.2), where $T_{\mu\nu}$ is the stress-energy tensor, then due to the interaction between the spin-0 particle and the source, the theory predicts either a wrong gravitational potential or a wrong light bending angle around the source[3]. The wrong prediction is there as long as the graviton mass is not rigorously zero, even if it is infinitesimally small, and this problem is called the vDVZ discontinuity.

Not long after the vDVZ discontinuity was revealed, Vainshtein argued that, when the graviton mass approaches zero, non-linearities become dominant, so the linearized approximation is not suitable for the consideration of massless limit [11]. In other words, the vDVZ discontinuity is an artifact of the linearization, and hence we may cure it by introducing interactions.

However, many other problems arise after introducing interactions. For instance, in 1972 Boulware and Deser found that the non-linear massive gravity theory can actually contain 6 physical degrees of freedom, in which the one extra degree of freedom is a ghost [12]. It is referred to as the Boulware-Deser ghost, and it exists in general when introducing non-linear terms into the Fierz-Pauli theory [13]. Despite of the general difficulty, in specific situations there is still hope that ghost-free models can be built [14, 15], but such models may contain modes that are propagating faster than the speed of light, which violates causality (see e.g. [16]).

1.3 Higher-Derivative Gravity in 3D

Since adding mass terms to general relativity can cause so many problems, it is natural to think about alternative ways of modification. One alternative way is to add correction terms with higher-order derivatives, which leads to a new kind of massive gravity theories. For instance, schematically one can write down a fourth-order derivative action like[4]

$$S = \int d^D x \sqrt{-g} \left\{ R + \frac{1}{m^2} \left(a R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + b R^{\mu\nu} R_{\mu\nu} + c R^2 \right) \right\} , \qquad (1.3.1)$$

[3]Here by "wrong" I mean the discrepancy in the prediction between the massless limit of the massive theory and the the genuine massless theory whose prediction has been proven by astronomical observations. Whether it is a wrong potential or a wrong light bending angle depends on a rescaling of the Newton's constant.

[4]Theoretically, gravity as a geometry of spacetime can also live in other dimensions. Here this schematic action is given in generic D-dimensional spacetime.

where a, b and c are parameters waiting to be tuned. On the contrary to the second-order derivative theory, here in (1.3.1), roughly speaking, it is the Einstein-Hilbert term that serves as the mass term of the massive graviton, and the correction terms are kinetic terms. It is important to notice that (1.3.1) usually contains both massive and massless gravitons. The way (1.3.1) reduces to general relativity is to take $m \to \infty$ limit, which makes the massive graviton infinitely heavy.

An unfortunate fact about (1.3.1) is that the massive graviton is a ghost, and if we flip the overall sign of the action, then the massive graviton is no longer a ghost but at the cost that the massless graviton turns into a ghost [17].

However, there is one way to circumvent this problem: to go to the 3-dimensional spacetime. A well-known fact is that general relativity (i.e. the Einstein-Hilbert action) in 3D does not describe any propagating graviton. Similarly, (1.3.1) in 3D contains no massless graviton. This implies that we can flip the overall sign of the action to save the massive graviton. So 3D higher-derivative gravity theories can be interesting toy models for studying massive gravity.

1.3.1 New Massive Gravity

A 3D higher-derivative gravity model of the type (1.3.1) was proposed by Bergshoeff, Hohm and Townsend in 2009 [18]. It is known as the *New Massive Gravity*, whose action reads[5]

$$S = \frac{M_{\mathrm{P}}}{2} \int d^3x \sqrt{-g} \left\{ -R + \frac{1}{m^2} \left(R^{\mu\nu} R_{\mu\nu} - \frac{3}{8} R^2 \right) \right\} , \tag{1.3.2}$$

where M_{P} is the 3D Planck mass and m is the graviton mass. An action of the type (1.3.1) in general contains a massive spin-0 mode, which is suppressed here in (1.3.2) by tuning the last coefficient to $-\frac{3}{8}$.

The action (1.3.2) contains a massive graviton with two propagating degrees of freedom. Its overall sign is chosen in such a way that the massive graviton is not a ghost. The Einstein-Hilbert term in (1.3.2) has a "wrong" sign, but this does not produce any ghost due to the absence of the massless graviton in 3D. Furthermore, it has been shown that this theory is free of Boulware-Deser ghost [19]. So in short, this is a ghost-free interacting theory describing a massive graviton that carries two degrees of freedom.

In many second-order derivative theories like (1.2.2), the gauge symmetry is broken by the mass terms[6]. However here in the New Massive Gravity theory,

[5] According to [18], a cosmological-constant parameter can be introduced into the model, and how the mass parameter m may change the effective cosmological constant has been discussed in the same paper. In this thesis, I mainly focus on models on the flat background, so here I do not introduce the cosmological constant.

[6] As a side remark, one can restore the symmetry by adding extra terms of some new fields

since all the higher-derivative terms are constructed by covariant tensors, the diffeomorphism symmetry is preserved. This is an advantage of higher-derivative theories over lower-derivative theories, and it has been exploited for higher-spin studies, which I will present in this thesis.

1.3.2 Topologically Massive Gravity

Long before New Massive Gravity was proposed, there was another 3D higher-derivative theory which contains up to third-order derivatives, and whose graviton carries only one degree of freedom. It was proposed by Deser, Jakiw and Templeton in [20], and the action reads

$$S = \frac{M_{\mathrm{P}}}{2} \int d^3x \sqrt{-g} \left\{ -R \pm \frac{1}{2m} \epsilon^{\mu\nu\rho} \Gamma^{\alpha}_{\mu\beta} \left(\partial_\nu \Gamma^{\beta}_{\alpha\rho} + \frac{2}{3} \Gamma^{\beta}_{\nu\gamma} \Gamma^{\gamma}_{\rho\alpha} \right) \right\} , \tag{1.3.3}$$

where Γ is the Levi-Civita connection. It is named as the *Topologically Massive Gravity*, because of its Chern-Simons-like higher-derivative terms. Again, the Einstein-Hilbert term has a "wrong" sign for the same reason as that in the New Massive Gravity. Moreover, Topologically Massive Gravity is also invariant under the diffeomorphism transformation (up to total derivatives).

The New Massive Gravity theory is invariant under the parity transformation, but the Topologically Massive Gravity is not. The former contains two modes with opposite helicities that are interchanged by a parity transformation;[7] the latter contains only one mode, and the $\pm$ sign in (1.3.3) represents two different choices of helicities. A relation between these two models will be discussed in the next chapter with more details.

1.4 Outline of This Thesis

In this thesis I will present several extensions of the New Massive Gravity theory and the Topologically Massive Gravity theory.

Chapter 2 will be the extension to bosonic higher-spin gauge theories.

We know that string theory predicts the existence of a tower of higher-spin particles, and higher-spin studies attracted a lot of attention in recent years. How-

(named Stückelberg fields), whose gauge transformation cancels with that of the mass terms, but I will not discuss this trick here.

[7] Considering a massive particle in its rest frame in 3D spacetime, it is easy to imagine that its angular momentum (spin) can take "up" and "down" directions, as there are clockwise and counterclockwise rotations on a two-dimensional plane. The parity transformation in 3D spacetime flips one spatial dimension, which interchanges these two states. More details will be discussed in the thesis.

ever, the difficulty of constructing interactions for higher spins still remains. Usually when writing down interaction terms of a theory, we follow the guidance of its gauge symmetry. Therefore, we think before being able to construct interactions it would be useful to first create gauge symmetries.

Early in 1978, Fronsdal developed in [21] the gauge theory of massless integer higher-spin fields (with only second-order derivatives). In his model, the spin-s theory has a gauge symmetry parameterized by a symmetric rank-$(s-1)$ tensor which has to be traceless. Moreover by adding mass terms to the theory, i.e. going to the Fierz-Pauli theory, the gauge symmetry is broken.

In Chapter 2, I will show that, inspired by the New/Topologically Massive Gravity (in particular their way of preserving the diffeomorphism symmetry), one can construct in 3D at the linearized level another kind of higher-spin gauge theory, which is more elegant in the sense that it is a massive theory, but it has a gauge symmetry and moreover its gauge parameter, unlike the one in Fronsdal's theory, is free of the traceless constraint.

Chapter 3 will be the discussion on the possibility to extend the 3D theories in Chapter 2 to higher dimensions.

Since the 3D higher-derivative gravity and its bosonic higher-spin extension have many nice properties, it would be nice, if they could be extended to higher dimensions, so that they might be applicable to the real world. However, as stated below (1.3.1), ghosty massless modes very often pop up in higher dimensions, so it won't be an easy task. In Chapter 3, I will show that only in some specific situations can we extend our 3D models (currently only at the linearized level) to higher dimensions. The method used is to introduce tensors of mixed symmetry to avoid the massless modes.

Chapter 4 will be the extensions to fermions and supergravity in 3D.

In this chapter, I will show that the theories presented in Chapter 2 can be extended to include fermions. In particular, we can construct such models for gravitini, and then combine them with graviton models into supergravity theories. The discussion will further be extended to supersymmetric New Massive Gravity, focusing on whether we can construct an equivalent lower-derivative version of the model in [22]. The supersymmetry rules involved in the first part of this discussion close only on-shell. In the second part of this discussion, at the linearized level, the way to derive the off-shell version of this model is presented.

Furthermore, there will be several appendices. Appendix A is the collection of conventions that are not specified in the main body of the thesis. Appendix B is the action for FP spin-s in arbitrary dimensions. Appendix C gives the generalized Cotton tensor in 3D. Appendix D presents useful information of Young tableaux and symmetrizers. Appendix E explains the parity transformation of fermions in

3D, in comparison with the 4D case.

Chapter 2

Extension to 3D Bosonic Higher Spins

The linearized New Massive Gravity (NMG) and Topologically Massive Gravity (TMG) theories can be extended to 3D bosonic massive higher-spin theories with gauge symmetries[1]. They are spin-2 examples of a much larger framework of models.

For pedagogical reasons, this chapter will not start from NMG or TMG. Instead, the starting point will be the Fierz-Pauli (FP) theory, which is the standard free theory for massive higher spins and contains only up to second-order derivatives. We will first introduce the FP equations and the methodology of constructing actions for them; then we will show how to perform in 3D a "boosting up derivative" procedure to create a gauge symmetry in the FP theory, which results into a NMG-like higher-derivative theory. Explicit actions for the the NMG-like models will be shown up to spin-4, and the problem of ghosts will be examined in general. Afterwards, the TMG-like models will be introduced as ghost-free alternative models, and the relation between the NMG-like and the TMG-like models will be explained.

2.1 Fierz-Pauli Theory

2.1.1 Fierz-Pauli Equations

The theory of higher-spin particles was initiated by Fierz and Pauli [8]. In the FP theory a bosonic massive spin-s particle (in the D-dimensional Minkowski spacetime) is represented by a symmetric rank-s tensor field, say $\varphi_{\mu_1\cdots\mu_s} \equiv \varphi_{(\mu_1\cdots\mu_s)}$, that satisfies the divergenceless and traceless conditions:

$$\partial^{\mu_1}\varphi_{\mu_1\cdots\mu_s} = 0 \,, \tag{2.1.1}$$

$$\eta^{\mu_1\mu_2}\varphi_{\mu_1\cdots\mu_s} = 0 \,. \tag{2.1.2}$$

[1]Only bosonic theories are presented in Chapter 2 and 3. In these two chapters, the spin number is always understood as an integer, unless otherwise specified.

Currently we are only able to extend the theories at the linearized level. All discussions in chapter 2 and 3 will be only on free theories on the flat background, unless otherwise specified.

By imposing these two conditions, all redundant degrees of freedom are suppressed. One can check, in 4 dimensions for instance, if we subtract the number of independent constraints imposed by these conditions from the number of total independent components of the symmetric tensor, we obtain $2s+1$, which is indeed the right number of propagating degrees of freedom carried by a spin-s particle.

Furthermore, in the FP theory, the propagation of the massive spin-s particle is described by the Klein-Gordon equation:

$$\left(\Box - m^2\right) \varphi_{\mu_1 \cdots \mu_s} = 0 \ . \tag{2.1.3}$$

The set of equations (2.1.3), (2.1.1) and (2.1.2) are called the Fierz-Pauli equations.

The original FP theory was constructed in 4D, but it is also applicable in other dimensions. For instance, in 3D the above FP equations are formally the same. However, one should be aware of an underlying difference between 3D and 4D: if we perform the counting, we find that the 3D FP equations for an arbitrary spin-s particle carry only 2 propagating degrees of freedom.

This difference can be understood from the perspective of group theory. The little group for 4D massive particles is SO(3), whose irreducible representation labelled by the spin quantum number s carries $2s+1$ degrees of freedom. However in the 3D massive case, the little group is SO(2), which is analogous to the 4D massless case. Just like the photon in 4D, which has two opposite helicities, in 3D the massive field φ under the conditions (2.1.1) and (2.1.2) also contains two helicities, each of which corresponds to an irrep of SO(2).

2.1.2 Construction of Actions

Actions up to spin-4 were constructed in [23], and later the generic action for arbitrary spins was constructed in [24]. In this subsection, the basic methodology of constructing actions for the FP theory will be briefly reviewed, and later in this thesis, the actions for higher-derivative models will be constructed in a similar manner.

Spin-1 We begin with the simplest example: FP spin-1, which is called the Proca model, whose equations of motion are

$$\left(\Box - m^2\right) \varphi_\mu = 0 \ , \ \partial^\mu \varphi_\mu = 0 \ . \tag{2.1.4}$$

Now the aim is to combine these equations into one action, and the method is to first write down an ansatz for the action

$$S_{\text{FP spin-1}} = \int d^D x \left\{ \frac{1}{2} \varphi^\mu \left(\Box - m^2\right) \varphi_\mu + \frac{a}{2} \left(\partial^\mu \varphi_\mu\right)^2 \right\} \ , \tag{2.1.5}$$

where the first term is the leading Klein-Gordon term, to which we should add all possible quadratic terms with the order of derivatives not higher than the leading term (here for spin-1 $(\partial^\mu \varphi_\mu)^2$ is the only possible term). a is a free parameter, and we shall tune it in such a way that both equations in (2.1.4) can be derived.

From (2.1.5), the variation $\frac{\delta S}{\delta \varphi^\mu} = 0$ leads to an equation of motion of rank-1 (i.e. with one free index):

$$\left(\Box - m^2\right)\varphi_\mu - a\partial_\mu\left(\partial^\nu \varphi_\nu\right) = 0 \; . \tag{2.1.6}$$

By taking the divergence of (2.1.6), i.e. $\partial^\mu\left(\frac{\delta S}{\delta\varphi^\mu}\right) = 0$, we obtain another equation of rank-0:

$$\left(\Box - a\Box - m^2\right)\partial^\mu \varphi_\mu = 0 \; . \tag{2.1.7}$$

Then it is obvious that we should set the parameter $a = 1$ in order to suppress the d'Alembertians, and thus (2.1.7) leads to the divergenceless condition in (2.1.4). Moreover, by substituting this divergenceless condition back into (2.1.6), the Klein-Gordon equation in (2.1.4) is also derived. Therefore in conclusion, (2.1.5) with $a = 1$ is the action for (2.1.4).

Spin-2 The method for spin-2 is similar. Again, one can write down for the FP equations

$$\left(\Box - m^2\right)\varphi_{\mu\nu} = 0 \; , \; \partial^\mu \varphi_{\mu\nu} = 0, \; \eta^{\mu\nu}\varphi_{\mu\nu} = 0 \tag{2.1.8}$$

an action ansatz with the leading Klein-Gordon term and all quadratic terms with no more than second-order derivatives:

$$\begin{aligned} S_{\text{FP spin-2}} \;\; &= \;\; \int d^D x \left\{ \frac{1}{2}\varphi^{\mu\nu}\left(\Box - m^2\right)\varphi_{\mu\nu} + \frac{1}{2}\varphi\left(a\Box - bm^2\right)\varphi \right. \\ &\qquad \left. + \frac{1}{2}c\varphi^{\mu\nu}\partial_\mu\partial^\rho\varphi_{\rho\nu} + d\varphi^{\mu\nu}\partial_\mu\partial_\nu\varphi \right\} \; , \end{aligned} \tag{2.1.9}$$

where $\varphi_{\mu\nu}$ is a symmetric tensor, $\varphi \equiv \eta^{\mu\nu}\varphi_{\mu\nu}$, and a, b, c and d are parameters to be tuned.

Note that the field $\varphi_{\mu\nu}$ off-shell is traceful, and the traceless condition of $\varphi_{\mu\nu}$ will only be derived on-shell as an equation of motion. In the literature [23] and [24] however, the fundamental field of the action is traceless, which is merely a difference in the choice of bases. One can do a field redefinition

$$\varphi_{\mu\nu} = \tilde{\varphi}_{\mu\nu} + \alpha\eta_{\mu\nu}\varphi' \; , \tag{2.1.10}$$

which splits $\varphi_{\mu\nu}$ into a traceless tensor field $\tilde{\varphi}_{\mu\nu}$ and a scalar field φ' (the coefficient α can be any non-zero real number). In this way, (2.1.9) can be converted into an equivalent action constructed by traceless fundamental fields, in which the scalar field will turn out to be an auxiliary field that vanishes on-shell.

Rank	Equations of motion		Results expected by tuning
2	$\left(\frac{\delta S}{\delta\varphi^{\mu\nu}}\right)=0$	$\Rightarrow$	$\left(\Box-m^2\right)\varphi_{\mu\nu}=0$
1	$\partial^\mu\left(\frac{\delta S}{\delta\varphi^{\mu\nu}}\right)=0$	$\Rightarrow$	$\partial^\mu\varphi_{\mu\nu}=0$
0	$\partial^\mu\partial^\nu\left(\frac{\delta S}{\delta\varphi^{\mu\nu}}\right)=0$ $\eta^{\mu\nu}\left(\frac{\delta S}{\delta\varphi^{\mu\nu}}\right)=0$	$\Rightarrow$	$\partial^\mu\partial^\nu\varphi_{\mu\nu}=0$ $\eta^{\mu\nu}\varphi_{\mu\nu}=0$

Table 2.1: Tuning parameters in the FP spin-2 action

From (2.1.9) one can derive an equation of motion of rank-2, and by taking divergences and the trace of this equation, one obtains a tower of equations of different ranks as indicated at the left side of Table 2.1. The right side of the table shows that, in a similar way to the tower of equations, one can write down a tower of field structures that shall be proved to vanish after the tuning (except the one at the top replaced by the Klein-Gordon equation). Just like what was done in (2.1.7), for each rank the tuning aims at algebraically deriving the vanishing of the field structures from the set of equations by doing substitutions and making necessary cancellations of d'Alembertians.

To illustrate what is exactly meant by Table 2.1, the detailed tuning procedure is shown as the following:

Step 1: one can derive directly from the ansatz an equation of motion of rank-2

$$\left(\Box-m^2\right)\varphi_{\mu\nu}+\eta_{\mu\nu}\left(a\Box-bm^2\right)\varphi+c\partial_{(\mu}\partial^\rho\varphi_{\nu)\rho}+d\partial_\mu\partial_\nu\varphi+d\eta_{\mu\nu}\partial^\rho\partial^\sigma\varphi_{\rho\sigma}=0\ . \tag{2.1.11}$$

Obviously, if we assume all lower-rank field structures ($\partial^\mu\varphi_{\mu\nu}$, $\partial^\mu\partial^\nu\varphi_{\mu\nu}$ and $\eta^{\mu\nu}\varphi_{\mu\nu}$) are zero, then this equation reduces to the Klein-Gordon equation, so in the following steps, our task is to tune the parameters so that these field structures indeed vanish.

Step 2: now by taking the divergence of (2.1.11), one can derive an equation of rank-1

$$\begin{aligned}\left(\Box-m^2\right)\partial^\mu\varphi_{\mu\nu}+\left(a\Box-bm^2\right)\partial_\nu\varphi+\frac{1}{2}c\Box\partial^\rho\varphi_{\nu\rho}+\frac{1}{2}c\partial_\nu\partial^\mu\partial^\rho\varphi_{\mu\rho}&\\+d\Box\partial_\nu\varphi+d\partial_\nu\partial^\rho\partial^\sigma\varphi_{\rho\sigma}=0\ .&\end{aligned} \tag{2.1.12}$$

One can see in this equation, if we assume all lower-rank field structures ($\partial^\mu\partial^\nu\varphi_{\mu\nu}$ and $\eta^{\mu\nu}\varphi_{\mu\nu}$) are zero, and set $c=-2$, then the d'Alembertians in front of $\partial^\mu\varphi_{\mu\nu}$ are suppressed and thus this equation reduces to $\partial^\mu\varphi_{\mu\nu}=0$. Therefore, all we have to do in the next step is to tune the rest of the parameters in order to make sure that the two rank-0 field structures indeed vanish.

Step 3: by taking the double divergence and the trace of the original rank-2 equation of motion (2.1.11), one can derive two rank-0 equations ($c = -2$ has been substituted):

$$\left(\Box - m^2\right)\partial^\mu\partial^\nu\varphi_{\mu\nu} + \left(a\Box^2 - bm^2\Box\right)\varphi - 2\Box\partial^\mu\partial^\nu\varphi_{\mu\nu} + d\Box^2\varphi + d\Box\partial^\mu\partial^\nu\varphi_{\mu\nu} = 0 \ , \tag{2.1.13}$$

$$\left(\Box - m^2\right)\varphi + D\left(a\Box - bm^2\right)\varphi - 2\partial^\mu\partial^\nu\varphi_{\mu\nu} + d\Box\varphi + dD\partial^\mu\partial^\nu\varphi_{\mu\nu} = 0 \ . \tag{2.1.14}$$

In (2.1.14), the field structure $\partial^\mu\partial^\nu\varphi_{\mu\nu}$ does not have d'Alembertians in front, so we can solve this equation for $\partial^\mu\partial^\nu\varphi_{\mu\nu}$ (otherwise we should try to solve for other field structures or suppress some d'Alembertian by tuning the parameters):

$$\partial^\mu\partial^\nu\varphi_{\mu\nu} = -\frac{1}{-2+dD}\left[\left(\Box - m^2\right)\varphi + D\left(a\Box - bm^2\right)\varphi + d\Box\varphi\right] \ , \tag{2.1.15}$$

assuming $-2 + dD \neq 0$. Then one can use this solution to eliminate $\partial^\mu\partial^\nu\varphi_{\mu\nu}$ in (2.1.13), which gives

$$\begin{aligned} -\frac{1}{-2+dD}\Big[&\left(-1+2a+2d+d^2-aD-d^2D\right)\Box^2 \\ &-\left(2b+2d+aD+bD-2bD\right)m^2\Box + \left(1+bD\right)m^4\Big]\varphi = 0 \ . \end{aligned} \tag{2.1.16}$$

Now we can see that in order to derive $\eta^{\mu\nu}\varphi_{\mu\nu} = 0$, one needs to suppress the d'Alembertians by tuning

$$a = \frac{-1+2d+d^2-d^2D}{D-2} \quad \text{and} \quad b = \frac{-4d-D+4dD+d^2D-d^2D^2}{(D-2)^2} \ .$$

Furthermore, (2.1.15) shows that by doing this tuning $\partial^\mu\partial^\nu\varphi_{\mu\nu}$ also vanishes as a consequence of $\eta^{\mu\nu}\varphi_{\mu\nu}$ being zero.

With the above three steps, we have almost finished the tuning, except for the parameter d. It is not a surprise that one free parameter remains, because we always have the freedom to rescale the trace part of $\varphi_{\mu\nu}$.[2] By setting $d = 1$ the result is simplified as $a = -1$ and $b = -1$.[3] Thus the action becomes[4]

$$\begin{aligned} S_{\text{FP spin-2}} \ = \ \int d^Dx \, \Big\{&\frac{1}{2}\varphi^{\mu\nu}\left(\Box - m^2\right)\varphi_{\mu\nu} - \frac{1}{2}\varphi\left(\Box - m^2\right)\varphi \\ &-\varphi^{\mu\nu}\partial_\mu\partial^\rho\varphi_{\rho\nu} + \varphi^{\mu\nu}\partial_\mu\partial_\nu\varphi\Big\} \ , \end{aligned} \tag{2.1.17}$$

from which the Klein-Gordon equation and all vanishing field structures (including the divergenceless and traceless conditions) can be derived.

For simplicity of the discussion, in the rest of this thesis, when explaining the construction of actions, we will only show tables like Table 2.1 without giving too many details in the tuning procedure.

[2] To be more precise, (2.1.9) can always be transformed into an equivalent action by doing the field redefinition which replaces $\varphi_{\mu\nu}$ with $\varphi_{\mu\nu} + \beta\eta_{\mu\nu}\varphi$ (β may take any real value except $-\frac{1}{D}$).

[3] The choice of d should satisfy the assumption $-2 + dD \neq 0$, which we made in step 3. In this thesis we only discuss theories in $D \geqslant 3$, so the choice $d = 1$ is always fine.

[4] As a side remark, this action is the same as the linearized massive gravity action (1.2.2) by renaming the field.

Rank	Equations of motion		Results expected by tuning
3	$\left(\frac{\delta S}{\delta \varphi^{\mu\nu\rho}}\right) = 0$	$\Rightarrow$	$\left(\Box - m^2\right)\varphi_{\mu\nu\rho} = 0$
2	$\partial^\mu \left(\frac{\delta S}{\delta \varphi^{\mu\nu\rho}}\right) = 0$	$\Rightarrow$	$\partial^\mu \varphi_{\mu\nu\rho} = 0$
1	$\partial^\mu\partial^\nu \left(\frac{\delta S}{\delta \varphi^{\mu\nu\rho}}\right) = 0$ $\eta^{\mu\nu} \left(\frac{\delta S}{\delta \varphi^{\mu\nu\rho}}\right) = 0$	$\Rightarrow$	$\partial^\mu\partial^\nu \varphi_{\mu\nu\rho} = 0$ $\eta^{\mu\nu}\varphi_{\mu\nu\rho} = 0$
0	$\partial^\mu\partial^\nu\partial^\rho \left(\frac{\delta S}{\delta \varphi^{\mu\nu\rho}}\right) = 0$ $\eta^{\mu\nu}\partial^\rho \left(\frac{\delta S}{\delta \varphi^{\mu\nu\rho}}\right) = 0$	$\Rightarrow$	$\partial^\mu\partial^\nu\partial^\rho \varphi_{\mu\nu\rho} = 0$ $\eta^{\mu\nu}\partial^\rho\varphi_{\mu\nu\rho} = 0$

Table 2.2: Tuning parameters in the FP spin-3 action (no auxiliary field)

Spin-3 As explained under (2.1.10), the FP spin-2 action in the traceless basis needs an auxiliary field, which can be avoided by going to the traceful basis. However, for higher spins, auxiliary fields are inevitable even in the traceful basis. Let us look at FP spin-3 for example.

Similar to spin-1 and spin-2, one can first try an ansatz for spin-3 without auxiliary fields:

$$\begin{aligned} S_{\text{FP spin-3}} \overset{?}{=} \int d^D x \Big\{ & \frac{1}{2}\varphi_{\mu\nu\rho}\left(\Box - m^2\right)\varphi^{\mu\nu\rho} + \frac{1}{2}\varphi_\rho\left(a_1\Box - a_2 m^2\right)\varphi^\rho \\ & + \frac{1}{2}a_3\varphi^{\mu\nu\rho}\partial_\mu\partial^\sigma\varphi_{\sigma\nu\rho} + \frac{1}{2}a_4\varphi^\rho\partial_\rho\partial_\sigma\varphi^\sigma + 3\varphi^{\mu\nu\rho}\partial_\mu\partial_\nu\varphi_\rho \Big\} , \end{aligned} \tag{2.1.18}$$

where $\varphi_{\mu\nu\rho}$ is a symmetric traceful tensor, and $\varphi_\rho \equiv \eta^{\mu\nu}\varphi_{\mu\nu\rho}$. For simplicity of the discussion, the last coefficient has been fixed to eliminate the freedom of rescaling the trace part of $\varphi_{\mu\nu\rho}$. As shown in Table 2.2, we would expect, by tuning a_1, a_2, a_3 and a_4, for each rank the vanishing field structures at the right side of the table can be derived from equations on the left (assuming lower-rank field structures are vanishing).

However this is not possible. Here the details of the calculation are skipped, but one can check that by going through the rank-2 and rank-1 tuning, one has to set $a_3 = -3$ and $a_1 = -3, a_2 = -3$, respectively, which means for rank-0 only one parameter a_4 is available for the tuning. The rank-0 equations shown in Table 2.2 can be converted into:

$$\begin{aligned} & 3D\left(\partial^\mu\partial^\nu\partial^\rho\varphi_{\mu\nu\rho}\right) + \left\{\left[(d+2)\,a_4 - 3\,(D-1)\right]\Box + 3\,(D+1)\,m^2\right\}\left(\eta^{\mu\nu}\partial^\rho\varphi_{\mu\nu\rho}\right) = 0 , \\ & \left[(D-1)\,(2a_4+3)\,\Box^2 + \left(3D + (D+2)\,a_4\right)m^2\Box + 3\,(D+1)\,m^4\right]\left(\eta^{\mu\nu}\partial^\rho\varphi_{\mu\nu\rho}\right) = 0 . \end{aligned} \tag{2.1.19}$$

Here we expect in the second equation to suppress the first two terms with $\Box^2$ and $\Box$ operators in order to make $\eta^{\mu\nu}\partial^\rho\varphi_{\mu\nu\rho} = 0$ (then consequently we also have

Rank	Equations of motion		Results expected by tuning
0	$\partial^\mu\partial^\nu\partial^\rho\left(\frac{\delta S}{\delta\varphi^{\mu\nu\rho}}\right)=0$ $\eta^{\mu\nu}\partial^\rho\left(\frac{\delta S}{\delta\varphi^{\mu\nu\rho}}\right)=0$ $\left(\frac{\delta S}{\delta\pi}\right)=0$	$\Rightarrow$	$\partial^\mu\partial^\nu\partial^\rho\varphi_{\mu\nu\rho}=0$ $\eta^{\mu\nu}\partial^\rho\varphi_{\mu\nu\rho}=0$ $\pi=0$

Table 2.3: Tuning parameters in the FP spin-3 action with the auxiliary field at rank-0

$\partial^\mu\partial^\nu\partial^\rho\varphi_{\mu\nu\rho}=0$ due to the first equation), but with only one parameter a_4 we are not able to suppress two terms. That is the reason why we need auxiliary fields for help.

Now we add the following terms to (2.1.18):

$$\begin{aligned}S_{\text{FP spin-3}} \quad = \quad & \int d^Dx\,\Big\{\frac{1}{2}\varphi_{\mu\nu\rho}\left(\Box-m^2\right)\varphi^{\mu\nu\rho}+\frac{1}{2}\varphi_\rho\left(a_1\Box-a_2m^2\right)\varphi^\rho \\ & +\frac{1}{2}a_3\varphi^{\mu\nu\rho}\partial_\mu\partial^\sigma\varphi_{\sigma\nu\rho}+\frac{1}{2}a_4\varphi^\rho\partial_\rho\partial_\sigma\varphi^\sigma+3\varphi^{\mu\nu\rho}\partial_\mu\partial_\nu\varphi_\rho \\ & +m\pi(\eta^{\mu\nu}\partial^\rho\varphi_{\mu\nu\rho}) \\ & +\frac{1}{2}\pi\left(a_5\Box-a_6m^2\right)\pi\Big\}\end{aligned} \tag{2.1.20}$$

As shown in the third line of the above action, because our goal is to make $\eta^{\mu\nu}\partial^\rho\varphi_{\mu\nu\rho}$ vanish, intuitively we can try to couple it an auxiliary scalar field π, and expect π to act in some way like a Lagrange multiplier. However, by doing this we get extra terms of π in the equations of motion of φ, so we must make sure that π is vanishing on-shell, which is the reason why we also add all possible quadratic terms of π (up to second-order derivatives for the FP theory) as shown in the last line of the above action.

Because π is a rank-0 field structure, it does not change the rank-3,2,1 part of Table 2.2. In the tuning process for higher ranks, we assume that all rank-0 field structures are vanishing, so the extra terms contributed by π does not change the above already-obtained value of a_1, a_2 and a_3. π only affects the tuning for rank-0, which is shown in Table 2.3. One can check that with two more parameters a_5 and a_6 in hand, after some manipulation of the equations at the left side of Table 2.3, it is possible to obtain all necessary cancellations of d'Alembertians, which leads to the results at the right side of the table. Details of this calculation are skipped here, and the resulting action is

$$\begin{aligned} S_{\text{FP spin-3}} \;=\; & \int d^D x \left\{ \frac{1}{2}\varphi_{\mu\nu\rho}\left(\Box - m^2\right)\varphi^{\mu\nu\rho} - \frac{3}{2}\varphi_\rho\left(\Box - m^2\right)\varphi^\rho \right. \\ & -\frac{3}{2}\varphi^{\mu\nu\rho}\partial_\mu\partial^\sigma\varphi_{\sigma\nu\rho} - \frac{3}{4}\varphi^\rho\partial_\rho\partial_\sigma\varphi^\sigma + 3\varphi^{\mu\nu\rho}\partial_\mu\partial_\nu\varphi_\rho \\ & +m\pi\partial_\rho\varphi^\rho \\ & \left. -\frac{1}{2}\pi\left(\frac{4\left(D-1\right)}{3\left(D-2\right)}\Box - \frac{2D^2}{\left(D-2\right)^2}m^2\right)\pi \right\} . \end{aligned} \tag{2.1.21}$$

The action for generic spin-s FP equations was given by Singh and Hagen in [24] using traceless tensors. They did the calculation in 4D, but their result can also be extended to other dimensions (see Appendix B).

To summarize, the necessity of auxiliary fields is a general feature of higher-spin actions. Usually we just couple the auxiliary field with the field structure which we want to set to zero, and furthermore we also add quadratic terms of the auxiliary fields themselves. Later in this thesis, when discussing higher-derivative theories, also some actions with auxiliary fields will be introduced, and we will skip their process of construction, because the way of making ansatzes and tuning parameters for them will be similar to the above-presented examples.

2.2 Boosting Up the Derivatives

In the following part of this chapter, we focus on 3D spacetime. As already mentioned in Section 1.4, the FP theory does not contain any gauge symmetry, unless we add extra Stückelberg fields to it. In this section, we will see that in 3D there is another way of adding gauge symmetry to the FP theory, namely "boosting up the derivatives", which is inspired by the linearized New Massive Gravity.

2.2.1 New Massive Gravity and FP Spin-2 in 3D

If we linearize the NMG theory around the flat background, i.e. do a perturbative expansion around its Minkowski solution and take its lowest-order approximation, the NMG equations of motion become [18]:

$$\left(\Box - m^2\right) G_{\mu\nu}\left(h\right) = 0 \ , \ \eta^{\mu\nu} G_{\mu\nu}\left(h\right) = 0 \ , \tag{2.2.1}$$

which are a Klein-Gordon equation and a traceless condition of the linearized Einstein tensor $G_{\mu\nu}\left(h\right)$:

$$G_{\mu_1\mu_2}\left(h\right) = \varepsilon_{\mu_1}{}^{\nu_1\rho_1}\varepsilon_{\mu_2}{}^{\nu_2\rho_2}\partial_{\nu_1}\partial_{\nu_2}h_{\rho_1\rho_2} \ . \tag{2.2.2}$$

The field $h_{\mu\nu}$ is the graviton ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$), which is a symmetric and traceful tensor.

From (2.2.2) obviously one can see that $G_{\mu_1\mu_2}(h)$ is symmetric and divergenceless. Furthermore, $G_{\mu_1\mu_2}(h)$ is invariant under the transformation:

$$\delta h_{\rho_1\rho_2} = \partial_{(\rho_1}\xi_{\rho_2)} \ , \tag{2.2.3}$$

which is the linearized diffeomorphism symmetry. Thus the equations (2.2.1) also contain such a gauge symmetry.

It is very interesting to compare the NMG equations (2.2.1) with the FP spin-2 equations (2.1.8). One can see that they are actually two equivalent sets of equations that each describe a massive spin-2 particle with two propagating degrees of freedom, and they are related by the Einstein tensor:

$$\varphi_{\mu\nu} = G_{\mu\nu}(h) \ . \tag{2.2.4}$$

The above formula can be seen as the solution to the divergenceless condition in (2.1.8), and one can substitute this solution into the Klein-Gordon equation and the traceless condition in (2.1.8), which exactly leads to (2.2.1). In this way, starting from the FP spin-2 equations, by "boosting up the derivatives", we can derive the linearized NMG as an equivalent set of equations, which contains a gauge symmetry by virtue of its higher order of derivatives.

Inspired by the relation between these two theories, we find a way to create a gauge symmetry in the generic spin-s FP theory in 3D.

2.2.2 Generic Spin-s NMG-Like Models

Now we apply the "boosting up derivative" method, as a tool to create the gauge symmetry, to the generic spin-s situation [25].

For the spin-s FP equations in 3D

$$\left(\Box - m^2\right)\varphi_{\mu_1\cdots\mu_s} = 0 \ , \ \partial^{\mu_1}\varphi_{\mu_1\cdots\mu_s} = 0 \ , \ \eta^{\mu_1\mu_2}\varphi_{\mu_1\cdots\mu_s} = 0 \ , \tag{2.2.5}$$

one can always solve the divergenceless condition by

$$\varphi_{\mu_1\cdots\mu_s} = G_{\mu_1\mu_2\cdots\mu_s}(h) \ , \tag{2.2.6}$$

where the G is a "generalized Einstein tensor" that contains s derivatives[5]:

$$G_{\mu_1\mu_2\cdots\mu_s}(h) = \varepsilon_{\mu_1}{}^{\nu_1\rho_1}\cdots\varepsilon_{\mu_s}{}^{\nu_s\rho_s}\partial_{\nu_1}\cdots\partial_{\nu_s}h_{\rho_1\cdots\rho_s} \ , \tag{2.2.7}$$

[5] The expression of s exterior derivatives hitting h (without ε's) can be understood as a generalization of the linearized Riemann tensor. See [26] for a detailed discussion on the spin-3 generalization, but note that the definition of the generalized Einstein tensor therein is different from ours.

where h is a rank-s symmetric and traceful tensor. Then by substituting the solution into the remaining part of FP equations, one obtains a spin-s NMG-like model:

$$\left(\Box - m^2\right) G_{\mu_1\mu_2\cdots\mu_s}\left(h\right) = 0 \ , \ \eta^{\mu_1\mu_2} G_{\mu_1\mu_2\cdots\mu_s}\left(h\right) = 0 \ . \tag{2.2.8}$$

Due to the higher-derivative construction of the generalized Einstein tensor, this model possesses a gauge symmetry

$$\delta h_{\rho_1\rho_2\cdots\rho_s} = \partial_{(\rho_1}\xi_{\rho_2\cdots\rho_s)} \ , \tag{2.2.9}$$

where the gauge parameter field ξ is a symmetric and traceful tensor of rank $s-1$.

In other words, the linearized NMG equations actually belongs to a general framework of models that are equivalent to the 3D FP equations and that contain gauge symmetries.

2.2.3 Construction of Actions

It is natural to ask whether we can integrate (2.2.8) into an action. It turns out that no auxiliary field is needed up to spin-3, but for higher spins auxiliary fields are necessary. We have explicitly constructed the actions up to spin-4, which will be shown in the following.

Actions for spin-1,2,3 without auxiliary fields

It turns out that the actions for spin-1, 2 and 3 can be easily constructed without auxiliary fields [25]. It is convenient to exploit the generalized Cotton tensor $C_{\mu_1\cdots\mu_s}\left(h\right)$. In our definition it is symmetric in its indices, and it is constructed by $2s-1$ derivatives hitting $h_{\mu_1\cdots\mu_s}$ (see Appendix C for details). For $s = 1, 2, 3$, it can be expressed as

$$\begin{aligned} C_{\mu_1}\left(h\right) &= G_{\mu_1}\left(h\right) \ , && (2.2.10)\\ C_{\mu_1\mu_2}\left(h\right) &= \varepsilon_{(\mu_1|}{}^{\nu_1\rho_1}\partial_{\nu_1}G_{\rho_1|\mu_2)}\left(h\right) \ , && (2.2.11)\\ C_{\mu_1\mu_2\mu_3}\left(h\right) &= \varepsilon_{(\mu_1}{}^{\nu_1\rho_1}\varepsilon_{\mu_2|}{}^{\nu_2\rho_2}\partial_{\nu_1}\partial_{\nu_2}S_{\rho_1\rho_2|\mu_3)}\left(h\right) \ , && (2.2.12)\end{aligned}$$

respectively, where S is the Schouten tensor:

$$S_{\mu_1\mu_2\mu_3}\left(h\right) = G_{\mu_1\mu_2\mu_3}\left(h\right) - \frac{3}{4}\eta_{(\mu_1\mu_2}G_{\mu_3)\nu}{}^{\nu}\left(h\right) \ . \tag{2.2.13}$$

The generalized Cotton tensor is useful, because it is by definition both divergenceless and traceless. Suppose in an action we use the generalized Cotton tensor to build the kinetic term (e.g. schematically $h\ C\left(h\right)$), and use the generalized Einstein tensor to build the mass term (schematically $m^2 h\ G\left(h\right)$), then naively we would expect, by taking the trace of its equation of motion, the former term drops and the latter term remains, which leads to the traceless condition of the Einstein tensor. After that, we would expect the equation of motion reduces to the Klein-Gordon equation of the Einstein tensor by substituting the traceless condition.

As shown in (2.2.8), for spin-1, 2 and 3 the kinetic terms contain 3rd, 4th and 5th-order derivatives, respectively. Then in the Lagrangian naively we can try to construct the kinetic terms as $h^{\mu_1}\Box C_{\mu_1}(h)$, $h^{\mu_1\mu_2}\varepsilon_{\mu_1}{}^{\nu_1\rho_1}\partial_{\nu_1}C_{\rho_1\mu_2}(h)$ and $h^{\mu_1\mu_2\mu_3}C_{\mu_1\mu_2\mu_3}(h)$, respectively, so that they have the right orders of derivatives. Furthermore, one can prove that these terms are self-adjoint.[6]

It turns out that this is indeed a right way of construction.

The action for spin-1 is simply

$$S_{\text{NMG spin-1}} = \int d^3x \left\{ \frac{1}{2} h^{\mu_1}\Box C_{\mu_1}(h) - \frac{m^2}{2} h^{\mu_1} G_{\mu_1}(h) \right\} , \tag{2.2.14}$$

from which the Klein-Gordon equation of the generalized Einstein tensor can be directly derived, and for spin-1 there is no traceless condition.

The action for spin-2 is

$$S_{\text{NMG spin-2}} = \int d^3x \left\{ \frac{1}{2} h^{\mu_1\mu_2}\varepsilon_{\mu_1}{}^{\nu_1\rho_1}\partial_{\nu_1}C_{\rho_1\mu_2}(h) - \frac{m^2}{2} h^{\mu_1\mu_2} G_{\mu_1\mu_2}(h) \right\} , \tag{2.2.15}$$

which is exactly the linearized version of the New Massive Gravity action.

The action for spin-3 is

$$S_{\text{NMG spin-3}} = \int d^3x \left\{ \frac{1}{2} h^{\mu_1\mu_2\mu_3} C_{\mu_1\mu_2\mu_3}(h) - \frac{m^2}{2} h^{\mu_1\mu_2\mu_3} G_{\mu_1\mu_2\mu_3}(h) \right\} . \tag{2.2.16}$$

From this action one can derive a rank-3 equation of motion

$$C_{\mu_1\mu_2\mu_3}(h) - m^2 G_{\mu_1\mu_2\mu_3}(h) = 0 . \tag{2.2.17}$$

By taking the trace of this equation one obtains the traceless condition

$$\eta^{\mu_1\mu_2} G_{\mu_1\mu_2\mu_3}(h) = 0 , \tag{2.2.18}$$

which can be substituted back into the rank-3 equation, resulting into the Klein-Gordon equation

$$\left(\Box - m^2\right) G_{\mu_1\mu_2\mu_3}(h) = 0 . \tag{2.2.19}$$

However, when $s \geq 4$ the above simple way of construction does not work, because we expect the kinetic term to have derivatives of the order $s+2$, but the

[6] Being self-adjoint means varying either of the two h's gives the same thing up to total derivative terms, e.g. $\delta h^{\mu_1}\Box C_{\mu_1}(h) = h^{\mu_1}\Box C_{\mu_1}(\delta h) +$ total derivatives.

generalized Cotton tensor has the order $2s-1$, which is always larger than $s+2$ for $s \geq 4$. Therefore, in this situation we need auxiliary fields for help. In the following, we illustrate how this is done for the spin-4 case.

A spin-4 action with auxiliary fields

The spin-4 NMG-like equations of motion are

$$\left(\Box - m^2\right) G_{\mu\nu\rho\sigma}(h) = 0 \quad , \quad G^{\rm tr}_{\mu\nu}(h) = 0 \quad , \tag{2.2.20}$$

where

$$G_{\mu\nu\rho\sigma}(h) = \varepsilon_\mu{}^{\tau\alpha}\varepsilon_\nu{}^{\eta\beta}\varepsilon_\rho{}^{\xi\gamma}\varepsilon_\sigma{}^{\zeta\delta}\partial_\tau\partial_\eta\partial_\xi\partial_\zeta\, h_{\alpha\beta\gamma\delta} \quad , \quad G^{\rm tr}_{\mu\nu}(h) = \eta^{\rho\sigma}G_{\mu\nu\rho\sigma}(h) \quad . \tag{2.2.21}$$

The construction of the action is similar to the method introduced in Section 2.1.2: we first write down the leading Klein-Gordon term $h^{\mu\nu\rho\sigma}\left(\Box - m^2\right) G_{\mu\nu\rho\sigma}(h)$,[7] then because we also want to include the traceless condition, we couple the trace of the generalized Einstein tensor with an auxiliary field, say $\pi^{\mu\nu}G^{\rm tr}_{\mu\nu}(h)$, and furthermore we add another part of the action that purely depends on auxiliary fields in order to make sure they are indeed on-shell vanishing. By writing down an ansatz and tuning the parameters, we obtain

$$\begin{aligned} S_{\rm NMG\ spin\text{-}4} \quad = \quad & \int d^3x\ \Big\{\frac{1}{2m^6}h^{\mu\nu\rho\sigma}\left(\Box - m^2\right) G_{\mu\nu\rho\sigma}\left(h\right) \\ & +\frac{1}{m^4}\pi^{\mu\nu}G^{\rm tr}_{\mu\nu}\left(h\right) \\ & -\frac{1}{2m^2}\pi^{\mu\nu}G_{\mu\nu}(\pi) - \frac{1}{2}\left(\pi^{\mu\nu}\pi_{\mu\nu} - \pi^2\right) \\ & +\phi\pi + \frac{13}{12}\phi^2 + \frac{1}{12m^2}\phi\Box\phi\Big\} \quad , \end{aligned} \tag{2.2.22}$$

where we have introduced two auxiliary fields, a symmetric and traceful tensor $\pi_{\mu\nu}$ ($G_{\mu\nu}(\pi) = \varepsilon_\mu{}^{\tau\rho}\varepsilon_\nu{}^{\eta\sigma}\partial_\tau\partial_\eta\,\pi_{\rho\sigma}$, $\pi \equiv \eta^{\mu\nu}\pi_{\mu\nu}$) and a scalar field ϕ [27].

The main idea of the tuning process, or, the way to show that (2.2.20) can be derived from (2.2.22) is indicated by Table 2.4, which is similar to the previous tables for FP actions. Again, for each rank, after properly tuning the parameters, the vanishing field structures at the right side of the table should be derivable from the set of equations on the left, under the assumption that all lower-rank field structures are zero. The only difference from the FP actions is that here we treat the generalized Einstein tensor, instead of the gauge field h, as a basic building block of the vanishing field structures. There is only one trivial equation of rank-3,

[7] One can easily prove that this term is self-adjoint, which means by varying h it will give the Klein-Gordon operator acting on the generalized Einstein tensor.

Rank	**Equations of motion**		**Results expected by tuning**
4	$\left(\frac{\delta S}{\delta h^{\mu\nu\rho\sigma}}\right)=0$	$\Rightarrow$	$\left(\Box-m^2\right)G_{\mu\nu\rho\sigma}\left(h\right)=0$
3	$\partial^{\mu}\left(\frac{\delta S}{h^{\mu\nu\rho\sigma}}\right)=0$ gives $0=0$		$\partial^{\mu}G_{\mu\nu\rho\sigma}\left(h\right)$ vanishes by definition
2	$\eta^{\mu\nu}\left(\frac{\delta S}{\delta h^{\mu\nu\rho\sigma}}\right)=0$ $\left(\frac{\delta S}{\delta \pi^{\mu\nu}}\right)=0$	$\Rightarrow$	$\eta^{\mu\nu}G_{\mu\nu\rho\sigma}\left(h\right)=0$ $\pi_{\mu\nu}=0$
1	$\partial^{\mu}\left(\frac{\delta S}{\delta \pi^{\mu\nu}}\right)=0$	$\Rightarrow$	$\partial^{\mu}\pi_{\mu\nu}=0$
0	$\eta^{\mu\nu}\eta^{\rho\sigma}\left(\frac{\delta S}{\delta h^{\mu\nu\rho\sigma}}\right)=0$ $\partial^{\mu}\partial^{\nu}\left(\frac{\delta S}{\delta \pi^{\mu\nu}}\right)=0$ $\eta^{\mu\nu}\left(\frac{\delta S}{\delta \pi^{\mu\nu}}\right)=0$ $\left(\frac{\delta S}{\delta \phi}\right)=0$	$\Rightarrow$	$\eta^{\mu\nu}\eta^{\rho\sigma}G_{\mu\nu\rho\sigma}\left(h\right)=0$ $\partial^{\mu}\partial^{\nu}\pi_{\mu\nu}=0$ $\eta^{\mu\nu}\pi_{\mu\nu}=0$ $\phi=0$

Table 2.4: Deriving NMG-like spin-4 equations of motion from the action

which is the divergence of the original rank-4 equation. It is simply $0=0$, but this does not give any problem, because the divergence of the generalized Einstein tensor trivially vanishes by its definition, i.e. we do not need to derive any vanishing field structure for rank-3.

2.3 Canonical Analysis and Ghosts

In this section, the method of "canonical analysis" (see e.g. [28], [22]), which is useful for counting propagating degrees of freedom and for identifying ghosts, will be presented. At the beginning as an example the NMG-like spin-4 model will be analysed in some details, then we will move on to the discussion of generic spin-s in 3D. The issue of ghosts will be explained and summarized.

2.3.1 Analysis of the NMG-Like Spin-4 Model

Analysis of the Equations of Motion

We shall begin the analysis by confirming that the equations (2.2.20) indeed propagate two massive modes. This analysis will be useful when we later turn to a similar analysis of the actions. As we focus on the canonical structure of the equations, we make a time/space split for the components of the various fields, setting $\mu=(0,i)$ where $i=1,2$. We may then choose a gauge such that[8]

$$\partial^{i}h_{i\mu\nu\rho}=0\ . \tag{2.3.1}$$

[8] The gauge transformation is $\delta h_{\sigma\mu\nu\rho}=\partial_{(\sigma}\xi_{\mu\nu\rho)}$, then one can derive from $\partial^{i}\left(\delta h_{i\mu\nu\rho}\right)=0$ that $\nabla^2\xi_{\mu\nu\rho}=0$ ($\nabla^2=\partial^i\partial_i$), which means under this gauge-fixing condition ξ does not carry any degrees of freedom that may propagate.

In this gauge we may write the components of h in terms of five independent gauge-invariant fields $(\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4)$ as follows:

$$\begin{aligned} h_{0000} &= \frac{1}{(\nabla^2)^2}\varphi_0 \ , \qquad h_{000i} = \frac{1}{(\nabla^2)^2}\hat{\partial}_i\varphi_1 \ , \qquad h_{00ij} = \frac{1}{(\nabla^2)^2}\hat{\partial}_i\hat{\partial}_j\varphi_2 \ , \\ h_{0ijk} &= \frac{1}{(\nabla^2)^2}\hat{\partial}_i\hat{\partial}_j\hat{\partial}_k\varphi_3 \ , \qquad h_{ijkl} = \frac{1}{(\nabla^2)^2}\hat{\partial}_i\hat{\partial}_j\hat{\partial}_k\hat{\partial}_l\varphi_4 \ , \end{aligned} \tag{2.3.2}$$

where

$$\hat{\partial}_i = \varepsilon_{0i}{}^j\partial_j \ . \tag{2.3.3}$$

Note that we permit space non-locality, since this does not affect the canonical structure.[9] Substitution into (2.2.21) gives

$$\begin{aligned} G_{0000}(h) &= (\nabla^2)^2\varphi_4 \, , \qquad G_{000i}(h) = \nabla^2\left(\hat{\partial}_i\varphi_3 + \partial_i\dot{\varphi}_4\right) \ , \\ G_{00ij}(h) &= \left(\hat{\partial}_i\hat{\partial}_j\varphi_2 + 2\hat{\partial}_{(i}\partial_{j)}\dot{\varphi}_3 + \partial_i\partial_j\ddot{\varphi}_4\right) \ , \\ G_{0ijk}(h) &= \frac{1}{\nabla^2}\left[\hat{\partial}_i\hat{\partial}_j\hat{\partial}_k\varphi_1 + 3\hat{\partial}_{(i}\hat{\partial}_j\partial_{k)}\dot{\varphi}_2 + 3\hat{\partial}_{(i}\partial_j\partial_{k)}\ddot{\varphi}_3 + \partial_i\partial_j\partial_k\left(\partial_t^3\varphi_4\right)\right] \ , \\ G_{ijkl}(h) &= \frac{1}{(\nabla^2)^2}\left[\hat{\partial}_i\hat{\partial}_j\hat{\partial}_k\hat{\partial}_l\varphi_0 + 4\hat{\partial}_{(i}\hat{\partial}_j\hat{\partial}_k\partial_{l)}\dot{\varphi}_1 + 6\hat{\partial}_{(i}\hat{\partial}_j\partial_k\partial_{l)}\ddot{\varphi}_2 \right. \\ &\qquad \left. + 4\hat{\partial}_{(i}\partial_j\partial_k\partial_{l)}\left(\partial_t^3\varphi_3\right) + \partial_i\partial_j\partial_k\partial_l\left(\partial_t^4\varphi_4\right)\right] \ , \end{aligned} \tag{2.3.4}$$

and[10]

$$\begin{aligned} G^{\rm tr}_{00}(h) &= \nabla^2\left(\varphi_2 - \Box\varphi_4\right) \ , \\ G^{\rm tr}_{0i}(h) &= \hat{\partial}_i\left(\varphi_1 - \Box\varphi_3\right) + \partial_i\left(\dot{\varphi}_2 - \Box\dot{\varphi}_4\right) \ , \\ G^{\rm tr}_{ij}(h) &= \frac{1}{\nabla^2}\left[\hat{\partial}_i\hat{\partial}_j\left(\varphi_0 - \Box\varphi_2\right) + 2\hat{\partial}_{(i}\partial_{j)}\left(\dot{\varphi}_1 - \Box\dot{\varphi}_3\right) + \partial_i\partial_j\left(\ddot{\varphi}_2 - \Box\ddot{\varphi}_4\right)\right] \ . \end{aligned} \tag{2.3.5}$$

Using these results, the tensor equation $G^{\rm tr} = 0$ implies that

$$\varphi_0 = \Box^2\varphi_4 \, , \qquad \varphi_1 = \Box\varphi_3 \, , \qquad \varphi_2 = \Box\varphi_4 \, ,$$

which eliminates $(\varphi_0, \varphi_1, \varphi_2)$ as independent fields. Then the dynamical equation of (2.2.20) is equivalent to

$$\left(\Box - m^2\right)\varphi_3 = 0 \quad , \quad \left(\Box - m^2\right)\varphi_4 = 0 \ ,$$

which shows that as expected there are two propagating degrees of freedom of equal mass m.

[9] In other words, we treat ∇^2 as if it is a number. One can think of the field as a certain harmonic mode, then ∇^2 operating on the field gives a negative number $(ik)^2$, where k is the wavenumber. In such analysis we often use expressions with ∇^2 to rescale fields, and when doing this we must make sure it is invertible. For instance, $\frac{1}{\nabla^2+m^2}$ is not allowed because ∇^2 is negative and thus this expression has a singularity at $\nabla^2 = -m^2$.

[10] In the context of this canonical analysis, the $\Box$ operator is simply understood as a notation for $-\left(\partial_0\right)^2 + \nabla^2$, rather than being interpreted as a Lorentz-invariant object.

Analysis of the Action

Now we further examine whether the action (2.2.22) propagates two modes that are physical, rather than ghosts. To do this we first need to rewrite the action in terms of gauge-invariant variables only and then eliminate auxiliary fields to get an action for the propagating modes only. We have already seen how to write the gauge potential h in terms of the five gauge-invariant fields $(\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4)$. The auxiliary tensor $\pi_{\mu\nu}$ has six independent components, which we may write in terms of six independent fields $(\psi_0, \psi_1, \psi_2; \lambda_0, \lambda_1, \lambda_2)$ as follows:

$$\begin{aligned} \pi_{00} &= -\frac{1}{\nabla^2}\left(\psi_0 + 2\dot{\lambda}_0\right) , \qquad \pi_{0i} = -\frac{1}{\nabla^2}\left[\hat{\partial}_i\left(\psi_1 + \dot{\lambda}_1\right) + \partial_i\left(\lambda_0 + \dot{\lambda}_2\right)\right] , \\ \pi_{ij} &= -\frac{1}{\nabla^2}\left(\hat{\partial}_i\hat{\partial}_j\psi_2 + 2\hat{\partial}_{(i}\partial_{j)}\lambda_1 + 2\partial_i\partial_j\lambda_2\right) . \end{aligned} \tag{2.3.6}$$

The dependence on the variables $(\lambda_1, \lambda_2, \lambda_3)$ is that of a spin-2 gauge transformation, so the tensor $G_{\mu\nu}(\pi)$, which is invariant under such a transformation, depends only on the three variables (ψ_0, ψ_1, ψ_2). Specifically, substituting the above expressions for the components of $\pi_{\mu\nu}$ gives

$$\begin{aligned} G_{00}(\pi) &= -\nabla^2\psi_2 , \quad G_{0i}(\pi) = -\left(\hat{\partial}_i\psi_1 + \partial_i\dot{\psi}_2\right) , \\ G_{ij}(\pi) &= -\frac{1}{\nabla^2}\left(\hat{\partial}_i\hat{\partial}_j\psi_0 + 2\hat{\partial}_{(i}\partial_{j)}\dot{\psi}_1 + \partial_i\partial_j\ddot{\psi}_2\right) , \end{aligned} \tag{2.3.7}$$

and hence

$$\eta^{\mu\nu}G_{\mu\nu}(\pi) = -\left(\psi_0 - \Box\psi_2\right) .$$

We are now in a position to determine the form of the action in terms of the gauge-invariant variables. Direct substitution yields the result

$$S_{\text{NMG spin-4}} = \int d^3x \left\{\mathcal{L}_1 + \mathcal{L}_2\right\} ,$$

where

$$\begin{aligned} \mathcal{L}_1 &= \frac{4}{m^6}\varphi_1\left(\Box - m^2\right)\varphi_3 - \frac{2}{m^4}\psi_1\left(\varphi_1 - \Box\varphi_3\right) - \frac{1}{m^2}\psi_1^2 - \lambda_1^2 \\ &\quad - \left(\psi_1 + \dot{\lambda}_1\right)\frac{1}{\nabla^2}\left(\psi_1 + \dot{\lambda}_1\right) , \end{aligned} \tag{2.3.8}$$

which depends only on the four fields $(\varphi_1, \varphi_3; \psi_1; \lambda_1)$, and

$$\begin{aligned}\mathcal{L}_2 \;=\;& \frac{1}{m^6}\varphi_0\left(\Box - m^2\right)\varphi_4 + \frac{3}{m^6}\varphi_2\left(\Box - m^2\right)\varphi_2 \\ & -\frac{1}{m^4}\,\psi_0\left(\varphi_2 - \Box\varphi_4\right) - \frac{1}{m^4}\,\psi_2\left(\varphi_0 - \Box\varphi_2\right) - \frac{1}{m^2}\,\psi_0\psi_2 \\ & +2\lambda_2\psi_2 - \left(\psi_2 + 2\lambda_2\right)\frac{1}{\nabla^2}\left(\psi_0 + 2\dot{\lambda}_0\right) \\ & -\left(\lambda_0 + \dot{\lambda}_2\right)\frac{1}{\nabla^2}\left(\lambda_0 + \dot{\lambda}_2\right) + \phi\frac{1}{\nabla^2}\left(\psi_0 + 2\dot{\lambda}_0\right) \\ & -\phi\psi_2 - 2\phi\lambda_2 + \frac{13}{12}\phi^2 + \frac{1}{12m^2}\phi\Box\phi\,, \end{aligned} \tag{2.3.9}$$

which depends only on the remaining eight fields $(\varphi_0, \varphi_2, \varphi_4; \psi_0, \psi_2; \lambda_0, \lambda_2; \phi)$. We have already seen that the propagating fields are φ_3 and φ_4, so it must be that each of the two modes is propagated by each of these two parts of the action. We now aim to confirm this and to determine whether the propagated modes are physical or ghosts. A systematic analysis is possible but we give only the final results.

Discarding total derivatives, the Lagrangian $\mathcal{L}_1$ can be rewritten as

$$\mathcal{L}_1 = \frac{1}{m^8}\tilde{\varphi}_1\left(\Box - m^2\right)\tilde{\varphi}_1 + \tilde{\psi}_1^2 - \tilde{\lambda}_1^2 + \frac{1}{m^2}\tilde{\varphi}_3^2\,, \tag{2.3.10}$$

where

$$\begin{aligned}\tilde{\varphi}_1 \;&=\; \varphi_1 + m^2\varphi_3 + \frac{1}{2}m^2\psi_1\,, \qquad \tilde{\varphi}_3 = \varphi_3 - \frac{1}{m^2}\varphi_1 - \frac{1}{2}\psi_1\,, \\ \tilde{\psi}_1 \;&=\; \frac{1}{\sqrt{-\nabla^2}}\left[\left(1 + \frac{\nabla^2}{2m^2}\right)\psi_1 + \dot{\lambda}_1 + \frac{\nabla^2}{m^4}\varphi_1 - \frac{\nabla^2}{m^2}\varphi_3\right]\,, \\ \tilde{\lambda}_1 \;&=\; \lambda_1 + \frac{1}{m^4}\dot{\varphi}_1 + \frac{1}{2m^2}\dot{\psi}_1 - \frac{1}{m^2}\dot{\varphi}_3\,. \end{aligned} \tag{2.3.11}$$

Using these relations, the field equations of (2.3.10) can be shown to imply that $\psi_1 = \lambda_1 = 0$ and

$$\varphi_1 = \Box\varphi_3\,, \qquad \left(\Box - m^2\right)\varphi_3 = 0\,,$$

in agreement with our earlier conclusion that φ_3 is the only independent propagating field (in the original basis).

In a similar way, the Lagrangian $\mathcal{L}_2$ can be rewritten as

$$\mathcal{L}_2 = \frac{4}{m^6}\tilde{\varphi}_2\left(\Box - m^2\right)\tilde{\varphi}_2 - \tilde{\phi}\,\tilde{\varphi}_4 - \frac{1}{m^4}\tilde{\psi}_2\,\tilde{\varphi}_0 + \tilde{\lambda}_2\tilde{\psi}_0 + \tilde{\lambda}_0^2\,, \tag{2.3.12}$$

where

$$
\begin{aligned}
\tilde{\varphi}_0 &= \varphi_0 - m^2\varphi_2 + m^2\left(\Box + m^2\right)\psi_2 - m^2\Box\phi - \frac{7}{6}m^4\phi \ , \qquad \tilde{\varphi}_2 = \varphi_2 + \frac{1}{6}m^2\phi \ , \\
\tilde{\varphi}_4 &= \varphi_4 - \frac{1}{m^2}\varphi_2 - \frac{1}{6}\phi \ , \qquad \tilde{\psi}_0 = -\frac{1}{\nabla^2}\left(\psi_0 - \Box\psi_2 + \Box\phi\right) \ , \\
\tilde{\psi}_2 &= \psi_2 - \frac{1}{m^2}\left(\Box - m^2\right)\varphi_4 \ , \qquad \tilde{\lambda}_0 = \frac{1}{\sqrt{-\nabla^2}}\left(\lambda_0 - \dot{\psi}_2 - \dot{\lambda}_2 + \dot{\phi}\right) \ , \\
\tilde{\lambda}_2 &= 2\lambda_2 + \frac{\nabla^2}{m^4}\left(\varphi_2 - \Box\varphi_4\right) + \left(1 + \frac{\nabla^2}{m^2}\right)\psi_2 - \phi \ , \\
\tilde{\phi} &= \frac{7}{6}\phi - \frac{1}{6m^2}\Box\phi - \frac{1}{m^4}\left(\Box - m^2\right)\varphi_2 - \psi_2 \ .
\end{aligned} \tag{2.3.13}
$$

Using these relations the field equations of $\mathcal{L}_2$ can be shown to be equivalent to $\psi_0 = \psi_2 = \lambda_0 = \lambda_2 = \phi = 0$ and

$$
\varphi_0 = \Box^2\varphi_4 \quad , \quad \varphi_2 = \Box\varphi_4 \quad , \quad \left(\Box - m^2\right)\varphi_4 = 0 \quad ,
$$

again in agreement with our earlier conclusion that φ_4 is the only independent propagating field (in the original basis).

If we now recombine the two Lagrangians $\mathcal{L}_1$ and $\mathcal{L}_2$ and eliminate auxiliary fields we arrive at the Lagrangian

$$
\mathcal{L}_{\text{NMG spin-4}} = \frac{1}{m^8}\tilde{\varphi}_1\left(\Box - m^2\right)\tilde{\varphi}_1 + \frac{4}{m^6}\tilde{\varphi}_2\left(\Box - m^2\right)\tilde{\varphi}_2 \ .
$$

Observe that both terms have the same sign. This means that the overall sign can be chosen such that both modes are physical. In our conventions, the sign that we have chosen is precisely such that this is the case, so our spin-4 action is ghost-free.

2.3.2 Analysis of Generic NMG-Like Spin-s Models

Although the NMG-like equations of motion are equivalent to the ones in FP theory, their actions are not equivalent. The fact that the actions in FP theory are ghost-free does not mean their higher-derivative counterparts can also be ghost-free.

A very simple counterexample is the NMG-like spin-1 action (2.2.14). Similar to what is shown in the spin-4 case, for spin-1 one can do the gauge-fixing

$$
\partial^i h_i = 0 \ , \tag{2.3.14}
$$

and then decompose h as

$$
h_0 = \frac{1}{\sqrt{-\nabla^2}}\varphi_0 \quad , \quad h_i = \frac{1}{\sqrt{-\nabla^2}}\hat{\partial}_i\varphi_1 \ . \tag{2.3.15}
$$

By substituting this decomposition into the action (2.2.14), one obtains

$$
S_{\text{NMG spin-1}} = \int d^3x \left\{-\varphi_0\left(\Box - m^2\right)\varphi_1\right\} \ , \tag{2.3.16}
$$

which is formulated as an off-diagonal term. Then obviously if we do a field redefinition to diagonalize it, the two propagating modes will have different signs. Therefore, no matter how we choose the overall sign of the action, one of the two modes is a ghost.

In [25] it has been conjectured that for generic spin-s, if s is even, there exists a ghost-free NMG-like action, but if s is odd, the action contains one ghost. Although for spin higher than 4 no action has yet been explicitly constructed, we can still argue that, if the actions are built in a similar way as (2.2.22), the analysis of the actions will have a crucial difference between even spins and odd spins [27].

Recall that the auxiliary fields are needed to impose the traceless constraint on the generalized Einstein tensor. As a shortcut, we could construct an action for the dynamical equation alone, for which auxiliary fields are not needed, and then impose "by hand" the constraint equation. In other words, we consider the Lagrangian

$$\mathcal{L}_{\text{NMG spin-}s} = \frac{1}{2} h^{\mu_1 \cdots \mu_s} \left(\Box - m^2\right) G_{\mu_1 \cdots \mu_s}(h) \ . \tag{2.3.17}$$

To the field equations we must now add, "by hand", the trace-free constraint

$$G^{\text{tr}}_{\mu_1 \cdots \mu_{s-2}}(h) = 0 \,. \tag{2.3.18}$$

We now proceed to a canonical analysis of this Lagrangian, and the constraint, by setting

$$h_{i_1 i_2 \cdots i_t 0 \cdots 0} = \frac{1}{\left(-\nabla^2\right)^{s/2}} \hat{\partial}_{i_1} \cdots \hat{\partial}_{i_t} \varphi_t \ , \qquad (t = 0, \ldots, s). \tag{2.3.19}$$

It then follows, for $r = 0, \cdots, s$, that

$$G_{i_1 \cdots i_r 0 \cdots 0}(h) = (-1)^r \left(-\nabla^2\right)^{\frac{s}{2} - r} \sum_{p=0}^{r} \binom{r}{p} \hat{\partial}_{(i_1} \cdots \hat{\partial}_{i_p} \partial_{i_{p+1}} \cdots \partial_{i_r)} \partial_0^{r-p} \varphi_{s-p} \,, \tag{2.3.20}$$

and, for $r = 2, \cdots, s$, that

$$\begin{aligned} G^{\text{tr}}_{i_1 \cdots i_{r-2} 0 \cdots 0}(h) \quad &= \quad (-1)^{r+1} \left(-\nabla^2\right)^{\frac{s}{2} - r + 1} \sum_{p=0}^{r-2} \binom{r-2}{p} \times \\ & \qquad \hat{\partial}_{(i_1} \cdots \hat{\partial}_{i_p} \partial_{i_{p+1}} \cdots \partial_{i_{r-2})} \partial_0^{r-p-2} \left(\varphi_{s-p-2} - \Box \varphi_{s-p}\right) \ . \end{aligned} \tag{2.3.21}$$

Substituting into the Lagrangian (2.3.17) we obtain

$$\mathcal{L}_{\text{spin-}s} = \begin{cases} \frac{1}{2}\binom{s}{s/2}\varphi_{s/2}\left(\Box - m^2\right)\varphi_{s/2} & \\ \qquad + \sum_{t=0}^{(s/2)-1}\binom{s}{t}\varphi_t\left(\Box - m^2\right)\varphi_{s-t} & \text{even } s\,, \\ & \\ -\sum_{t=0}^{(s-1)/2}\binom{s}{t}\varphi_t\left(\Box - m^2\right)\varphi_{s-t} & \text{odd } s\,. \end{cases} \tag{2.3.22}$$

In either case, the equations of motion that follow from this Lagrangian are

$$\left(\Box - m^2\right)\varphi_t = 0\,, \qquad (t = 0, \cdots, s)\,. \tag{2.3.23}$$

To these equations we have to add the traceless condition (2.3.18), which is equivalent to

$$\varphi_{s-p-2} = \Box\varphi_{s-p}\,, \qquad (p = 0, \cdots, s-2)\,. \tag{2.3.24}$$

By combining (2.3.23) with (2.3.24) one finds, for $t \leq s$, that

$$\begin{aligned} \varphi_0 &= m^t\varphi_t\,, && t = 0, 2, 4, \cdots\,, \\ \varphi_1 &= m^{t-1}\varphi_t\,, && t = 1, 3, 5, \cdots\,. \end{aligned} \tag{2.3.25}$$

Next, one substitutes these equations into the Lagrangians of (2.3.22) in order to eliminate all fields other than φ_0 and φ_1. For even spin s the resulting Lagrangians contain only two terms: $\varphi_0\left(\Box - m^2\right)\varphi_0$ and $\varphi_1\left(\Box - m^2\right)\varphi_1$, both with the same (positive) sign. The even spin Lagrangians are therefore ghost-free. In contrast, the Lagrangians for odd spin contain only one off-diagonal term, which is proportional to $\varphi_0\left(\Box - m^2\right)\varphi_1$. In this case, therefore, one mode is physical and the other a ghost. Although this argument falls short of a proof that there is a ghost-free spin-s NMG-type action only for even s, we believe that it captures the essential difference between the even and odd spin cases.

2.3.3 Ghosts and Parity

Besides the calculation above, we have a less rigorous but more intuitive way of explaining why there are ghosts for odd spins and no ghosts for even spins. The crucial concept here is parity.[11] Here are two facts about parity. First, group

[11] The parity transformation by definition has always the determinant -1. In 4D we can define it as the reversal of either all three spatial directions or only one of them. In 3D there are two spatial dimensions, which is even, so we define it as the reversal of a single spatial dimension, e.g. the reversal of the 2nd spatial dimension, which transforms a vector as $V_\mu \to P_\mu{}^\nu V_\nu$, where $P_\mu{}^\nu = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$.

theoretically, when the parity is broken, the fundamental representation of O(2) may split into two irreps of SO(2). This implies that we may be able to identify the two propagating degrees of freedom in a NMG-like model as two modes with opposite helicities that are interchanged by a parity transformation. Second, the leading Klein-Gordon term of a NMG-like Lagrangian $h^{\mu_1\cdots\mu_s}\left(\Box - m^2\right)G_{\mu_1\cdots\mu_s}(h)$ is parity even for even spins, and parity odd for odd spins. This is because the Levi-Civita symbol is not invariant under the parity transformation:

$$\varepsilon_{\mu_1\mu_2\mu_3} \to P_{\mu_1}{}^{\nu_1}P_{\mu_2}{}^{\nu_2}P_{\mu_3}{}^{\nu_3}\varepsilon_{\nu_1\nu_2\nu_3} = \det(P)\,\varepsilon_{\mu_1\mu_2\mu_3} = -\varepsilon_{\mu_1\mu_2\mu_3}\ , \tag{2.3.26}$$

and hence the generalized Einstein tensor for spin-s, which contains s Levi-Civita symbols, acquires a factor $(-1)^s$.

Combining the above two facts, we can explain the issue of ghosts. Suppose by going through the canonical analysis, the Lagrangian ends up as

$$\begin{aligned}\mathcal{L}_{\text{NMG spin-}s} \quad \sim \quad & H^+\left(\Box - m^2\right)H^+ \pm H^-\left(\Box - m^2\right)H^- \\ & + (\text{decoupled auxiliary components})\ ,\end{aligned} \tag{2.3.27}$$

where H^+ and H^- stand for two modes with opposite helicities, whose terms may or may not have the same sign. Under the parity transformation, H^+ and H^- are interchanged into each other, and since we know that the leading part of the Lagrangian should acquire a factor $(-1)^s$, we can conclude that the two terms of opposite helicities have the same sign when s is even, and have different signs when s is odd. This is the reason why for even spins the NMG-like models are ghost-free by properly choosing an overall sign, but for odd spins there is always one ghost in the two modes.

Considering the above argument, it is natural to consider the possibility that we can discard one of the two helicity states, so that with only one physical mode we can always choose a good overall sign to make the model ghost-free. In fact it is possible, and for the spin-2 case, discarding one helicity state exactly leads to the linearized Topologically Massive Gravity. In the next section, this model will be discussed in some details.

2.4 The "$\sqrt{\text{FP}}$" and TMG-Like Models

In this section, we look at the theory with only one helicity state. In a similar way to the previous sections, the lower-derivative theory will be first introduced, then we will again "boost up derivatives".

2.4.1 "$\sqrt{\text{FP}}$" Equations

One important feature of the 3D theory is that under the divergenceless condition, the Klein-Gordon operator can be factorized [29, 25].

Spin-1

We first look at spin-1 FP equations as the simplest example:

$$\left(\Box - m^2\right) \varphi_\mu = 0 \ , \ \partial^\mu \varphi_\mu = 0 \ . \tag{2.4.1}$$

One can prove that under the divergenceless condition, the Klein-Gordon equation in 3D can be written in the following way

$$\left(\varepsilon_\mu{}^{\nu\rho} \partial_\nu \pm m \delta_\mu^\rho\right) \left(\varepsilon_\rho{}^{\sigma\tau} \partial_\sigma \mp m \delta_\rho^\tau\right) \varphi_\tau = 0 \ , \tag{2.4.2}$$

where $\left(\Box - m^2\right)$ has been factorized into two first-order differential operators. By dropping the first one of these two operators, we can write down two equations with only first-order derivatives

$$\begin{aligned} \varepsilon_\rho{}^{\sigma\tau} \partial_\sigma \varphi_\tau &= +m \varphi_\rho \ , \\ \varepsilon_\rho{}^{\sigma\tau} \partial_\sigma \varphi_\tau &= -m \varphi_\rho \ , \end{aligned} \tag{2.4.3}$$

so that any solution of either equation (or any linear combination of their solutions) is a solution to the Klein-Gordon equation. Moreover, one can easily check that the divergenceless condition is also encoded in each of these equations. So let us just call them spin-1 "$\sqrt{\text{FP}}$" equations.

Intuitively, we can imagine that the two equations in (2.4.3) may describe two vectors rotating in opposite directions on a plane. If we view $\varepsilon_\rho{}^{\sigma\tau} \partial_\sigma$ as a helicity operator, these two equations are actually describing two different helicity eigenstates. Also, one can clearly see that under the parity transformation, as shown in (2.3.26), the Levi-Civita symbol acquires a minus sign and thus the two equations in (2.4.3) are interchanged into each other. To summarize, the two propagating degrees of freedom in (2.4.1) have been identified as two opposite helicity states in a Lorentz-covariant way, i.e. they have been split into two first-order differential equations (2.4.3).

Spin-s

Now we look at how to take a "square root" of generic spin-s FP equations:

$$\left(\Box - m^2\right) \varphi_{\mu_1 \cdots \mu_s} = 0 \quad , \quad \partial^{\mu_1} \varphi_{\mu_1 \cdots \mu_s} = 0 \quad , \quad \eta^{\mu_1 \mu_2} \varphi_{\mu_1 \cdots \mu_s} = 0 \ . \tag{2.4.4}$$

Almost the same as the spin-1 situation, under the divergenceless condition, the Klein-Gordon equation can be written as

$$\left(\varepsilon_{\mu_1}{}^{\nu\rho}\partial_\nu \pm m\delta^\rho_{\mu_1}\right)\left(\varepsilon_\rho{}^{\sigma\tau}\partial_\sigma \mp m\delta^\tau_\rho\right)\varphi_{\tau\mu_2\cdots\mu_s} = 0 \;, \tag{2.4.5}$$

where only the first index of the field contracts with the one on the differential operator. Again, by dropping the first operator, one obtains the "$\sqrt{\text{FP}}$" as two first-order differential equations

$$\begin{aligned}\left(\varepsilon_{\mu_1}{}^{\sigma\tau}\partial_\sigma - m\delta^\tau_{\mu_1}\right)\varphi_{\tau\mu_2\cdots\mu_s} &= 0 \;,\\ \left(\varepsilon_{\mu_1}{}^{\sigma\tau}\partial_\sigma + m\delta^\tau_{\mu_1}\right)\varphi_{\tau\mu_2\cdots\mu_s} &= 0 \;,\end{aligned} \tag{2.4.6}$$

which separately describe two opposite helicity states that are interchanged by a parity transformation.

One can check that from either equation in (2.4.6) both the divergenceless and traceless conditions can be derived, and this relies on the fact that the index μ_1 is not symmetrized with other μ's. However, for convenience of constructing actions, it is useful to symmetrize these free indices, since they are likely to be contracted with the symmetrized indices of the fundamental field.[12] Then in this case, we still need to explicitly impose the two subsidiary conditions. In other words, (2.4.6) is equivalent to the following set of equations:[13]

$$\begin{aligned}\left(\varepsilon_{(\mu_1|}{}^{\sigma\tau}\partial_\sigma \mp m\delta^\tau_{(\mu_1|}\right)\varphi_{\tau|\mu_2\cdots\mu_s)} &= 0 \;,\\ \partial^{\mu_1}\varphi_{\mu_1\cdots\mu_s} &= 0 \;,\\ \eta^{\mu_1\mu_2}\varphi_{\mu_1\cdots\mu_s} &= 0 \;.\end{aligned} \tag{2.4.7}$$

2.4.2 Boosting Up the Derivatives: TMG-Like Models

Like what has been done in Section 2.2 for the FP theory, we can also boost up the derivatives for the $\sqrt{\text{FP}}$ theory. Again, we can solve the divergenceless condition by using the generalized Einstein tensor as shown in (2.2.6), and substitute the solution into (2.4.6) or (2.4.7), which gives

$$\left(\varepsilon_{\mu_1}{}^{\sigma\tau}\partial_\sigma - \mu\delta^\tau_{\mu_1}\right)G_{\tau\mu_2\cdots\mu_s}(h) = 0 \;, \tag{2.4.8}$$

[12] In principle it is possible to construct actions for $\sqrt{\text{FP}}$ equations using the same methodology introduced before, but this will not be discussed in this thesis. Actions for the higher-derivative version of these equations will be introduced later.

[13] In order to prove the equivalence, one needs to antisymmetrize in (2.4.6) the three indices on the Levi-Civita symbol with another μ on the field, and then apply the Schouten identity (i.e. antisymmetrizing four indices in 3D gives zero), in order to show that under the divergenceless and traceless conditions the only surviving components are the ones obtained by symmetrizing all μ's.

or equivalently

$$\begin{aligned} \left(\varepsilon_{(\mu_1|}{}^{\sigma\tau}\partial_\sigma - \mu\delta^\tau_{(\mu_1|}\right) G_{\tau|\mu_2\cdots\mu_s)}(h) &= 0 \, , \\ \eta^{\mu_1\mu_2} G_{\mu_1\cdots\mu_s}(h) &= 0 \, , \end{aligned} \tag{2.4.9}$$

where $\mu = m$ or $-m$, depending on which helicity state we choose. The spin-2 situation is exactly the linearized Topologically Massive Gravity, so in some sense the TMG is "$\sqrt{\text{NMG}}$", and by construction such TMG-like models contain the same gauge symmetries as the the NMG-like models do.

We would like to integrate equations in the TMG-like models into actions, and it turns out that for spin-1 and spin-2 auxiliary fields are not necessary. Once more, we exploit the generalized Cotton tensor to construct the kinetic terms:[14]

$$S_{\text{TMG spin-1}} = \int d^3x \left\{ \frac{1}{2} h^{\mu_1} \varepsilon_{\mu_1}{}^{\nu\rho} \partial_\nu C_\rho(h) - \frac{1}{2}\mu h^{\mu_1} G_{\mu_1}(h) \right\} \, , \tag{2.4.10}$$

$$S_{\text{TMG spin-2}} = \int d^3x \left\{ \frac{1}{2} h^{\mu_1\mu_2} C_{\mu_1\mu_2}(h) - \frac{1}{2}\mu h^{\mu_1\mu_2} G_{\mu_1\mu_2}(h) \right\} \, . \tag{2.4.11}$$

For spin-1 the expected rank-1 equation of motion can be directly derived from the action. For spin-2, one can derive from the action a rank-2 equation of motion:

$$C_{\mu_1\mu_2}(h) - \mu G_{\mu_1\mu_2}(h) = 0 \, ,$$

whose trace gives the traceless condition

$$\eta^{\mu_1\mu_2} G_{\mu_1\mu_2}(h) = 0 \, , \tag{2.4.12}$$

which can be substitute back into the rank-2 equation in order to derive

$$\left(\varepsilon_{(\mu_1|}{}^{\sigma\tau}\partial_\sigma - \mu\delta^\tau_{(\mu_1|}\right) G_{\tau|\mu_2)}(h) = 0 \, . \tag{2.4.13}$$

From spin-3 onwards, the generalized Cotton tensor cannot help, because for spin-s we expect the kinetic term to have derivatives of the order $s+1$, but the Cotton tensor has the order $2s-1$, which is always larger than $s+1$. Then we need auxiliary fields. The method is similar to the construction of the NMG-like spin-4 action (couple the trace G^{tr} with an auxiliary field and add other terms that purely consist of auxiliary fields), except that in the leading term, instead of the Klein-Gordon operator, we should use its factorized first-order operator.[15] The actions for spin-3 and spin-4 are explicitly given below.

[14] One can prove that all terms here are self-adjoint.

[15] This change of the operator in the leading term does not spoil the self-adjointness.

Rank	**Equations of motion**		**Results expected by tuning**
3	$\left(\frac{\delta S}{\delta h^{\mu\nu\rho}}\right)=0$	$\Rightarrow$	$\left(\epsilon_{(\mu|}{}^{\alpha\beta}\partial_\alpha-\mu\delta^\beta_{(\mu|}\right)G_{\beta|\nu\rho)}\left(h\right)=0$
2	$\partial^\mu\left(\frac{\delta S}{h^{\mu\nu\rho}}\right)=0$ gives $0=0$		$\partial^\mu G_{\mu\nu\rho}\left(h\right)$ vanishes by definition
1	$\eta^{\mu\nu}\left(\frac{\delta S}{\delta h^{\mu\nu\rho}}\right)=0$ $\left(\frac{\delta S}{\delta \pi^{\mu}}\right)=0$	$\Rightarrow$	$\eta^{\mu\nu}G_{\mu\nu\rho}\left(h\right)=0$ $\pi_\mu=0$
0	$\partial^\mu\left(\frac{\delta S}{\delta \pi^{\mu}}\right)=0$	$\Rightarrow$	$\partial^\mu\pi_\mu=0$

Table 2.5: Deriving TMG-like spin-3 equations of motion from the action

The spin-3 action can be constructed as

$$\begin{aligned} S_{\text{TMG spin-3}} \;=\; \int d^3x\, \Big\{ & \frac{1}{2}h^{\mu\nu\rho}\left(\epsilon_\mu{}^{\alpha\beta}\partial_\alpha-\mu\delta^\beta_\mu\right)G_{\beta\nu\rho}\left(h\right) \\ & +\mu\pi^\mu G^{\text{tr}}_\mu\left(h\right) \\ & -18\mu^4\pi^\mu\pi_\mu-6\mu^3\pi_\mu\varepsilon^{\mu\nu\rho}\partial_\nu\pi_\rho+\mu^2F^{\mu\nu}\left(\pi\right)F_{\mu\nu}\left(\pi\right)\Big\}\,, \end{aligned} \tag{2.4.14}$$

where $F_{\mu\nu}\left(\pi\right)=2\partial_{[\mu}\pi_{\nu]}$. Once again, as shown in Table 2.5, the parameters have been tuned rank by rank so that all the vanishing field structures can be derived.[16]

The spin-4 action can be given as

$$\begin{aligned} S \;=\; \int d^3x\, \Big\{ & \frac{1}{2}\mu h^{\mu\nu\rho\sigma}G_{\mu\nu\rho\sigma}\left(h\right)-\frac{1}{2}h^{\mu\nu\rho\sigma}\varepsilon_\mu{}^{\alpha\beta}\partial_\alpha G_{\beta\nu\rho\sigma}\left(h\right) \\ & +\mu\pi^{\mu\nu}G^{\text{tr}}_{\mu\nu}\left(h\right) \\ & +\mu^2\pi^{\mu\nu}C_{\mu\nu}\left(\pi\right)+2\mu^3\pi^{\mu\nu}G_{\mu\nu}\left(\pi\right)+4\mu^4\pi^{\mu\nu}\varepsilon_\mu{}^{\alpha\beta}\partial_\alpha\pi_{\beta\nu} \\ & +8\mu^5\left(\pi^{\mu\nu}\pi_{\mu\nu}-\pi^2\right)-\mu^4\phi^\mu\partial_\mu\pi+\mu^4\phi^\mu\partial^\alpha\pi_{\alpha\mu} \\ & -\frac{1}{16}\mu^4\phi^\mu\varepsilon_\mu{}^{\alpha\beta}\partial_\alpha\phi_\beta+\frac{1}{4}\mu^5\phi^\mu\phi_\mu\Big\}\,, \end{aligned} \tag{2.4.15}$$

where $\pi\equiv\eta^{\mu\nu}\pi_{\mu\nu}$, and $C_{\mu\nu}\left(\pi\right)$ and $G_{\mu\nu}\left(\pi\right)$ are the linearized Cotton and Einstein tensors of $\pi_{\mu\nu}$. The rank-by-rank process of tuning parameters and deriving vanishing fields structures is illustrated by Table 2.6.

Furthermore, one can perform a canonical analysis of the above TMG-like actions to check that they each contain one propagating degree of freedom, and consequently they are ghost-free by properly choosing their overall signs.

[16] There is a different way to generalize the linearized TMG to spin-3, which was given in [26], where the generalized Einstein tensor is defined with only second-order derivatives.

Rank	Equations of motion		Results expected by tuning
4	$\left(\frac{\delta S}{\delta h^{\mu\nu\rho\sigma}}\right)=0$	$\Rightarrow$	$\left(\epsilon_{(\mu\|}{}^{\alpha\beta}\partial_\alpha-\mu\delta^\beta_{(\mu\|}\right)G_{\beta\|\nu\rho\sigma)}\left(h\right)=0$
3	$\partial^\mu\left(\frac{\delta S}{h^{\mu\nu\rho\sigma}}\right)=0$ gives $0=0$		$\partial^\mu G_{\mu\nu\rho\sigma}\left(h\right)$ vanishes by definition
2	$\eta^{\mu\nu}\left(\frac{\delta S}{\delta h^{\mu\nu\rho\sigma}}\right)=0$ $\left(\frac{\delta S}{\delta\pi^{\mu\nu}}\right)=0$	$\Rightarrow$	$\eta^{\mu\nu}G_{\mu\nu\rho\sigma}\left(h\right)=0$ $\pi_{\mu\nu}=0$
1	$\partial^\mu\left(\frac{\delta S}{\delta\pi^{\mu\nu}}\right)=0$ $\left(\frac{\delta S}{\delta\phi^\mu}\right)=0$	$\Rightarrow$	$\partial^\mu\pi_{\mu\nu}=0$ $\phi_\mu=0$
0	$\eta^{\mu\nu}\eta^{\rho\sigma}\left(\frac{\delta S}{\delta h^{\mu\nu\rho\sigma}}\right)=0$ $\partial^\mu\partial^\nu\left(\frac{\delta S}{\delta\pi^{\mu\nu}}\right)=0$ $\eta^{\mu\nu}\left(\frac{\delta S}{\delta\pi^{\mu\nu}}\right)=0$ $\partial^\mu\left(\frac{\delta S}{\delta\phi^\mu}\right)=0$	$\Rightarrow$	$\eta^{\mu\nu}\eta^{\rho\sigma}G_{\mu\nu\rho\sigma}\left(h\right)=0$ $\partial^\mu\partial^\nu\pi_{\mu\nu}=0$ $\eta^{\mu\nu}\pi_{\mu\nu}=0$ $\partial^\mu\phi_\mu=0$

Table 2.6: Deriving TMG-like spin-4 equations of motion from the action

2.5 Solving Both Subsidiary Constraints

In the last section of this chapter, we briefly look at another way of boosting up the derivatives in 3D.

In the previous sections, as explained, we go to the higher-derivative theory always by solving the divergenceless condition, which is only one of the two subsidiary constraints. Actually, it is possible to simultaneously solve both subsidiary constraints [25, 27]. The way to solve them, is to use the generalized Cotton tensor (see Appendix C for its explicit definition).

To be precise, we can solve both

$$\partial^{\mu_1}\varphi_{\mu_1\cdots\mu_s}=0 \text{ and } \eta^{\mu_1\mu_2}\varphi_{\mu_1\cdots\mu_s}=0$$

by

$$\varphi_{\mu_1\cdots\mu_s}=C_{\mu_1\cdots\mu_s}\left(h\right)\ . \tag{2.5.1}$$

Then we can substitute this solution into the Klein-Gordon equation of φ:

$$\left(\Box-m^2\right)C_{\mu_1\cdots\mu_s}\left(h\right)=0\ , \tag{2.5.2}$$

or, into the $\sqrt{\text{FP}}$ of φ:[17]

$$\left(\varepsilon_{\mu_1}{}^{\sigma\tau}\partial_\sigma-\mu\delta^\tau_{\mu_1}\right)C_{\tau\mu_2\cdots\mu_s}\left(h\right)=0\ . \tag{2.5.3}$$

Just like the difference between NMG-like and TMG-like models, (2.5.2) describes two massive propagating degrees of freedom of opposite helicities, which are

[17]Since the generalized Cotton tensor is both divergenceless and traceless, one can prove that whether or not symmetrizing the free indices μ's gives equivalent equations.

interchanged by a parity transformation, while (2.5.3) describes only one massive propagating degree of freedom, and whether $\mu = m$ or $-m$ depends on which helicity state is chosen.

Both (2.5.2) and (2.5.3) possess the same gauge symmetry (2.2.9) as NMG/TMG-like models do, and furthermore they also contain the following gauge symmetry

$$\delta h_{\mu_1\cdots\mu_s} = \eta_{(\mu_1\mu_2}\Lambda_{\mu_3\cdots\mu_s)} \ , \tag{2.5.4}$$

which for the spin-2 case is the linearized conformal symmetry.

Equation (2.5.2) can be integrated into a simple action

$$S_{\text{KG Cotton}} = \int d^3x \frac{1}{2} h^{\mu_1\cdots\mu_s} \left(\Box - m^2\right) C_{\mu_1\cdots\mu_s}(h) \ . \tag{2.5.5}$$

Using the same reasoning as Section 2.3.3, due to the fact that the generalized Cotton tensor is always constructed by an odd number of Levi-Civita symbols, this action contains a ghost.

Equation (2.5.3) can also be integrated into a simple action

$$S_{\text{Sqrt-KG Cotton}} = \int d^3x \frac{1}{2} h^{\mu_1\cdots\mu_s} \left(\varepsilon_{\mu_1}{}^{\sigma\tau}\partial_\sigma - \mu\delta^\tau_{\mu_1}\right) C_{\tau\mu_2\cdots\mu_s}(h) \ , \tag{2.5.6}$$

and because it contains only one propagating degree of freedom, it can always be made ghost-free by choosing the right overall sign.

Chapter 3

Extension to Higher Dimensions

In the previous chapter, we have been focusing on 3D models. In this chapter, we discuss the possibility of extending NMG/TMG-like models to higher dimensions (at the linearized level). Once again, the discussion will be started in the context of the lower-derivative theory in higher dimensions, then we will discuss the possibility of boosting up the derivatives in order to obtain NMG-like models. Afterwards we will further discuss the possibility of factorizing the Klein-Gordon operator in higher dimensions, which may lead to TMG-like models. After all these general discussions, a concrete example in 7D will be presented in the last section of this chapter.

3.1 Tensors of Mixed Symmetry

Before discussing models in higher dimensions, the concept "spin" must be clarified.

In general dimensions, the type of a particle is defined by its irreducible representation of the little group. In 3D or 4D, where the little group for massive particles is SO(2) or SO(3), all inequivalent irreps can be represented by symmetric tensors of different ranks. Thus in 3D or 4D if we define "spin" to be the rank of the symmetric tensor, it is sufficient to label all different types of massive particles.

However, for massive particles in $D \geq 5$, whose little group is SO(4) or larger, only one spin number is not sufficient to label all types of particles. In this situation, the irreps of the little group are represented by not only symmetric tensors, but also antisymmetric tensors and various mixed symmetry tensors. Consequently we need Young tableaux, instead of spin, to give each type of them a unique label. Nevertheless, in this chapter the terminology "spin" will still be used to refer to the number of columns of a Young tableau.[1]

When we talk about irreps of the little group, we are actually talking about

[1] See Appendix D for details on Young tableaux, Young symmetrizers and their relation to spin. We acknowledge the frequent use of the software Cadabra [30, 31] to perform Young projection calculations.

traceless tensors with only spatial indices. However, the theories we would like to discuss are Lorentz-covariant. Therefore, we need to Lorentz-covariantize these tensors without changing the number of degrees of freedom they carry.

The FP theory has already told us how to Lorentz-covariantize symmetric tensors. We conjecture that the same method applies also to other types of tensors:
(1) We replace all spatial indices with spacetime indices while still keeping their symmetry properties;
(2) Because we would like to go from the little group to the Lorentz group, its natural to modify the traceless condition by replacing the Kronecker delta δ_{ij} with the spacetime metric $\eta_{\mu\nu}$;
(3) Because the tensor now carries more components than the irrep originally had, some extra constraint is needed. We would like such a constraint to be Lorentz-covariant, and we would like it to have no more than first-order derivatives, otherwise it might look like a dynamical equation rather than a constraint. Therefore, the only option seems to be the divergenceless condition, as already suggested by the FP theory.[2]

To illustrate this, we count, as an example, the degrees of freedom of the simplest mixed symmetry tensor in 4D. A tensor $T_{ij,k}$ which satisfies[3]

$$T_{ij,k} = \mathcal{Y}_{[2,1]} T_{ij,k} \quad \text{and} \quad \delta^{jk} T_{ij,k} = 0 \ , \tag{3.1.1}$$

as an irrep of the little group SO(3) carries 5 independent components.[4] Now we go to the group SO(1,3) by replacing all spatial indices with spacetime ones, and hence obtain the tensor $T_{\mu\nu,\rho}$, which satisfies

$$T_{\mu\nu,\rho} = \mathcal{Y}_{[2,1]} T_{\mu\nu,\rho} \quad \text{and} \quad \eta^{\nu\rho} T_{\mu\nu,\rho} = 0 \ , \tag{3.1.2}$$

carrying 16 independent components. We would like to impose the divergenceless condition[5]

$$\partial^\mu T_{\mu\nu,\rho} = 0 \tag{3.1.3}$$

in order to lower the degrees of freedom. One can check that $\partial^\mu T_{\mu\nu,\rho}$ is a non-symmetric traceless tensor, so it appears that (3.1.3) gives $4\times4-1 = 15$ independent

[2] We conjecture that the divergenceless condition always works well for the counting. Although we did not rigorously prove it, we have not yet found any counterexample.

[3] See Appendix D for details on notations. In this chapter, indices separated by commas are always understood as different sets of antisymmetrized indices.

[4] For the little group SO(3), a (reducible) traceful tensor of the type $\yng(2,1)$ has 8 independent components. The trace of it, which is the fundamental representation $\square$, has 3 independent components. Then a traceless tensor of the type $\yng(2,1)$ has $8-3=5$ independent components. One can do a similar counting that the same type of traceless tensor in the Lorentz group SO(1,3) carries 16 independent components.

[5] Note that because of the Young symmetry, the tensor satisfies $T_{[\mu\nu,\rho]} = 0$. Therefore, one can derive $\partial^\rho T_{\mu\nu,\rho} = 0$ from (3.1.3), i.e. the divergence hitting any one of the three indices gives zero.

constraints, but this is an over-counting, because the double divergence on the antisymmetric pair of indices vanishes by construction

$$\partial^\mu \partial^\nu T_{\mu\nu,\rho} \equiv 0 \ . \tag{3.1.4}$$

Therefore we must compensate 4 degrees of freedom. Thus the total number of degrees of freedom of $T_{\mu\nu,\rho}$ is $16 - 15 + 4 = 5$, which is exactly the same number as that of $T_{ij,k}$.

After such Lorentz covariantization, we can write down the generalized version of FP equations[6]

$$\left\{ \begin{array}{rcll} \left(\Box - m^2\right) T_{\cdots} & = & 0 & \\ \partial^{\cdot} T_{\cdots} & = & 0 & \text{(on all indices)} \\ \eta^{\cdot\cdot} T_{\cdots} & = & 0 & \text{(on all pairs of indices)} \end{array} \right. , \tag{3.1.5}$$

where T is a tensor of any allowed Young symmetry, and we use the Klein-Gordon equation to describe the free propagation. The divergenceless condition is always imposed on all indices and the traceless condition is always imposed on all pairs of indices. In the next section, we will discuss whether we can construct (linearized) NMG-like models that are equivalent to this set of generalized FP equations, like what we did in 3D.

3.2 Boosting Up the Derivatives

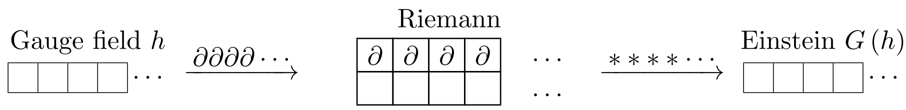
In the previous chapter, we derived in 3D the NMG-like models by solving the divergenceless condition in the FP equations using the generalized Einstein tensor, and we wonder whether a similar way of boosting up derivatives exists in higher dimensions.

Let us recall how we define the 3D generalized Einstein tensor (2.2.7). As shown in the following diagram, roughly speaking, it is defined in such a way that we first take the exterior derivative on every column of the Young tableau of the gauge field, resulting into the generalized Riemann tensor,[7] and then take the Hodge duality on every column of the Riemann tensor, which leads to what we call the generalized

[6] The generalized FP equations for some specific mixed-symmetry tensors have long been worked out. For instance, the equations for $\ydiagram{2,1}$ were constructed in [32], and the equations for $\ydiagram{3,1}$ were derived in [33] by doing dimensional reduction from the massless theory in $D+1$ dimensions. For more extensive discussions, see e.g. [34, 35, 36], where equations and Lagrangians for $\ydiagram{2,1}$, $\ydiagram{3,1}$ and $\ydiagram{2,2}$ are studied on the flat background, as well as (A)dS ones.

[7] One can check that for spin-2, the linearized Riemann tensor can be written as (up to an overall factor) taking an exterior derivative on each of the two indices of the graviton. If the generalized Riemann tensor vanishes, then the gauge field must be a pure gauge.

Einstein tensor.

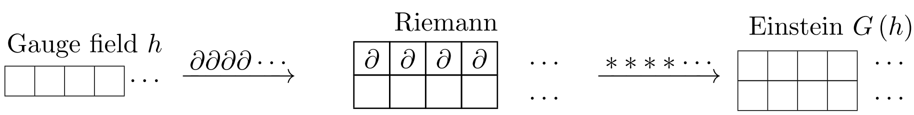


Note that there are two important facts in this way of construction.

First, in 3D, because the generalized Einstein tensor $G\,(h)$ has been constructed as the dual of the generalized Riemann tensor, setting $G\,(h)$ to zero means h is a pure gauge, which does not describe any propagating degrees of freedom. In this way $G\,(h) = 0$ gives only a trivial solution to the NMG-like equations of motion. This is important because otherwise the model would include massless propagating modes, which might be ghosts as mentioned in Section 1.3, and due to the existence of the massless modes, the model would not be equivalent to the FP theory.

Second, in 3D the generalized Einstein tensor $G\,(h)$ always lives in the same representation as the gauge field h (off-shell traceful and symmetric rank-s tensors), otherwise it would be difficult to integrate the equations of motion into an action.

In order to find direct extensions of the 3D NMG-like models in higher dimensions, we must be able to construct the generalized Einstein tensor with both properties mentioned above in $D > 3$. However, it is difficult in general. For instance, in 4D, if we use a totally symmetric gauge field and define the generalized Einstein tensor as the dual of the generalized Riemann tensor, then as shown in the diagram



$G\,(h)$ no longer lives in the same representation as h. On the other hand, if we define the generalized Einstein tensor in another way so that it can live in the same representation[8], then it carries less number of independent components than the generalized Riemann tensor, which means $G\,(h) = 0$ contains massless modes.

Only for gauge fields of specific representations (usually tensors of mixed symmetry) in specific dimensions can we define the generalized Einstein tensor to be the dual of the generalized Riemann tensor as well as in the same representation as the gauge field. Only in these situations, can we find a direct generalization of the 3D NMG-like models.

In these situations, we can use the generalized Einstein tensor to solve the

[8]See, for instance, the definition of the generalized Einstein tensor in [37], where it is basically constructed out of the traces of the Riemann tensor.

divergenceless condition in (3.1.5):[9]

$$T_{\cdots} = G_{\cdots}\left(h\right) \ , \tag{3.2.1}$$

where T, G and h have the same symmetry on their indices, and then substitute the solution into the other two equations in (3.1.5), in order to obtain the higher-derivative model

$$\begin{aligned} \left(\Box - m^2\right) G_{\cdots}\left(h\right) &= 0 \ , \\ \eta^{\cdot\cdot} G_{\cdots}\left(h\right) &= 0 \ , \end{aligned} \tag{3.2.2}$$

which is the NMG-like model in higher dimensions.

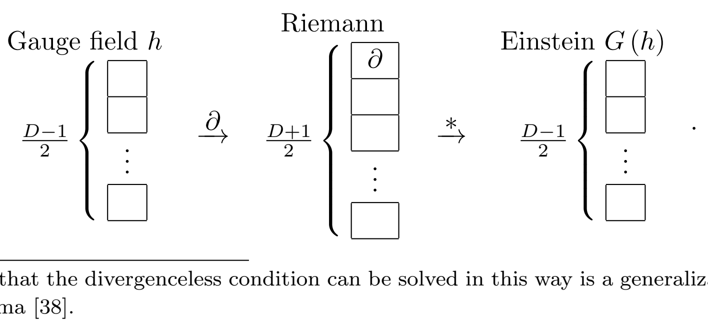
Similar to 3D, in higher dimensions the NMG-like models also have gauge symmetries. The generalized Einstein tensor by definition has a derivative in every column of its Young tableau, then because antisymmetrizing two derivatives gives zero, one can see that the gauge transformation rules are always parameterized by tensors with one index fewer than the gauge field has, and the missing index is carried by a derivative in the rule.

Furthermore, thanks to the fact that h and $G\left(h\right)$ live in the same representation, in some (but not all) situations , (3.2.2) can be integrated into an action with the leading Klein-Gordon term $h^{\cdots}\left(\Box - m^2\right) G_{\cdots}$ in the Lagrangian.

Now we discuss in detail exactly what representations are allowed to construct NMG-like models in higher dimensions. The situations for spin-1, spin-2 and higher spins differ from each other. In the following they will be discussed separately.

Spin-1

Considering the criterion that the dual of the Riemann tensor has to live in the same representation as the gauge field, for spin-1 the only allowed type of gauge field is represented by a single-column Young tableau of height $\frac{1}{2}(D-1)$ in an odd spacetime dimension D, as indicated in the diagram:



[9] The fact that the divergenceless condition can be solved in this way is a generalization of the Poincaré lemma [38].

Starting from the generalized FP equations for spin-1 (or, the higher-rank generalization of the Proca model)

$$\begin{aligned}\left(\Box - m^2\right) T_{\mu_1\cdots\mu_{(D-1)/2}} &= 0 \ ,\\ \partial^{\mu_1} T_{\mu_1\cdots\mu_{(D-1)/2}} &= 0 \ ,\end{aligned}$$

where T is a totally antisymmetric tensor, one can then solve the divergenceless condition[10]

$$T_{\mu_1\cdots\mu_{(D-1)/2}} = G_{\mu_1\cdots\mu_{(D-1)/2}}\left(h\right) = \varepsilon_{\mu_1\cdots\mu_{(D-1)/2}}{}^{\mu_{(D+1)/2}\cdots\mu_D}\partial_{\mu_{(D+1)/2}} h_{\mu_{(D+3)/2}\cdots\mu_D} \ , \tag{3.2.3}$$

and substitute the solution into the Klein-Gordon equation to derive the NMG-like model:

$$\left(\Box - m^2\right) G_{\mu_1\cdots\mu_{(D-1)/2}}\left(h\right) = 0 \ . \tag{3.2.4}$$

Obviously this model has the gauge symmetry

$$\delta h_{\mu_1\cdots\mu_{(D-1)/2}} = \partial_{[\mu_1}\xi_{\mu_2\cdots\mu_{(D-1)/2}]} \ . \tag{3.2.5}$$

Note that not all components of the gauge parameter field contribute to the transformation, i.e. there is a "gauge transformation" of the gauge parameter field itself: $\delta\xi_{\mu_2\cdots\mu_{(D-1)/2}} = \partial_{[\mu_2}\zeta_{\mu_3\cdots\mu_{(D-1)/2}]}$. This must be taken care of when we count the degrees of freedom and when we do the gauge-fixing. This also matters for spin-2 and higher spins. We will not further discuss it in this section, but in the last section of this chapter, for a specific 7D spin-2 example we will show some details.

In $D = 4k - 1$, where $k = 1, 2, \cdots$, (3.2.4) can be easily integrated into an action

$$S_{\text{NMG}\ \mathcal{Y}_{[2k-1]}} = \int d^{4k-1}x\ h^{\mu_1\cdots\mu_{2k-1}}\left(\Box - m^2\right) G_{\mu_1\cdots\mu_{2k-1}}\left(h\right) \ , \tag{3.2.6}$$

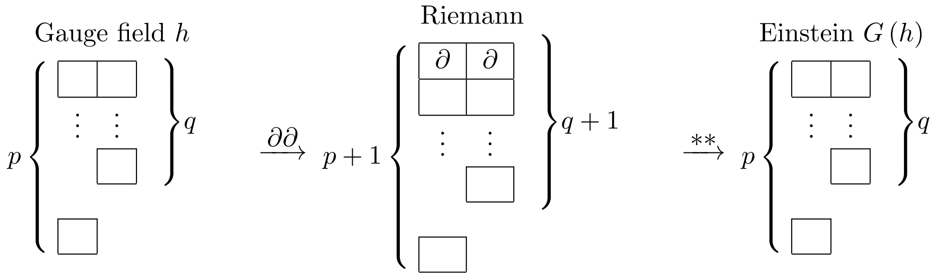
but in $D = 4k + 1$, there is not such an action, because in this situation one can prove that the Klein-Gordon term $h^{\mu_1\cdots\mu_{2k}}\left(\Box - m^2\right) G_{\mu_1\cdots\mu_{2k}}\left(h\right)$ is actually a total derivative.

Spin-2

For spin-2 in D dimensions, the allowed types of tensors always carry in total $D-1$ indices. As shown in the following diagram, denoting p and q ($p \geq q$) as the heights of the first and second columns of the Young tableau of the gauge field, if $p + q = D - 1$, then after taking exterior derivatives and Hodge dualities, the

[10]Here h and G carry antisymmetrized indices. In this chapter, indices that are not separated by commas are understood as antisymmetrized ones, which differs from the convention in the previous chapter.

height-p column ends up in a height-q column and vice versa.



$$(\; p+q = D-1 \;)$$

To write explicitly, the generalized Einstein tensor is constructed as

$$G_{\mu_1\cdots\mu_p,\nu_1\cdots\nu_q}(h) = \varepsilon_{\nu_1\cdots\nu_q}{}^{\alpha\rho_1\cdots\rho_p}\varepsilon_{\mu_1\cdots\mu_p}{}^{\beta\sigma_1\cdots\sigma_q}\partial_\alpha\partial_\beta h_{\rho_1\cdots\rho_p,\sigma_1\cdots\sigma_q} \; . \tag{3.2.7}$$

Then in the spin-2 generalized FP equations[11]

$$\begin{aligned} \left(\Box - m^2\right) T_{\mu_1\cdots\mu_p,\nu_1\cdots\nu_q} &= 0 \; , \\ \partial^{\mu_1} T_{\mu_1\cdots\mu_p,\nu_1\cdots\nu_q} &= 0 \; , \\ \eta^{\mu_1\nu_1} T_{\mu_1\cdots\mu_p,\nu_1\cdots\nu_q} &= 0 \; , \end{aligned} \tag{3.2.8}$$

one can solve the divergenceless condition by

$$T_{\mu_1\cdots\mu_p,\nu_1\cdots\nu_q} = G_{\mu_1\cdots\mu_p,\nu_1\cdots\nu_q}(h) \; , \tag{3.2.9}$$

which leads to the NMG-like model

$$\begin{aligned} \left(\Box - m^2\right) G_{\mu_1\cdots\mu_p,\nu_1\cdots\nu_q}(h) &= 0 \; , \\ \eta^{\mu_1\nu_1} G_{\mu_1\cdots\mu_p,\nu_1\cdots\nu_q}(h) &= 0 \; . \end{aligned} \tag{3.2.10}$$

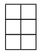
This model has the gauge symmetry

$$\delta h_{\mu_1\cdots\mu_p,\nu_1\cdots\nu_q} = \mathcal{Y}_{[p,q]}\left(\partial_{[\mu_p}\xi^1{}_{\mu_1\cdots\mu_{p-1}],\nu_1\cdots\nu_q} + \partial_{[\nu_q|}\xi^2{}_{\mu_1\cdots\mu_p,|\nu_1\cdots\nu_{q-1}]}\right) \tag{3.2.11}$$

for $p > q$, where ξ^1 and ξ^2 satisfy the Young symmetries $\mathcal{Y}_{[p-1,q]}$ and $\mathcal{Y}_{[p,q-1]}$, respectively. For $p = q$, because the two columns of the Young tableau of h are symmetric, using one tensor ξ of the type $\mathcal{Y}_{[p,p-1]}$ is sufficient to parameterize the transformation:

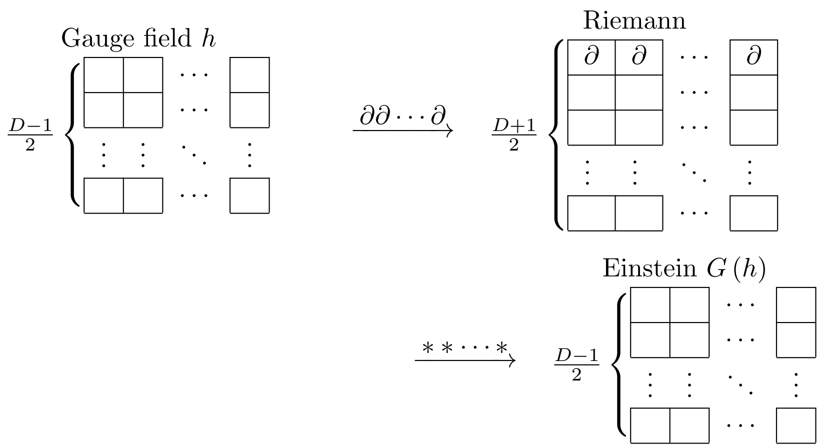
$$\delta h_{\mu_1\cdots\mu_p,\nu_1\cdots\nu_p} = \mathcal{Y}_{[p,p]}\left(\partial_{[\nu_p|}\xi_{\mu_1\cdots\mu_p,|\nu_1\cdots\nu_{p-1}]}\right) \; . \tag{3.2.12}$$

[11] T satisfies the Young symmetry $\mathcal{Y}_{[p,q]}$. Consequently, $T_{[\mu_1\cdots\mu_p,\nu_1]\cdots\nu_q} = 0$, which means $\partial^{\mu_1}T_{\mu_1\cdots\mu_p,\nu_1\cdots\nu_q} = 0$ implies that $\partial^{\nu_1}T_{\mu_1\cdots\mu_p,\nu_1\cdots\nu_q} = 0$.

We have not yet studied in general the possibility to integrate (3.2.10) into an action, but two specific examples have been studied in detail. One example is the gauge field of the type  in 4D. A NMG-like ghost-free action carrying 5 massive propagating degrees of freedom has been constructed using this type of gauge field in [39]. The other example is  in 7D, which is going to be discussed in the last section of this chapter.

Higher spins

For spin-3 or higher, besides the criterion that the dual of the Riemann tensor and the gauge field should have the same symmetry, there is another criterion which further eliminates a lot of representations: there should not be more than $D-1$ boxes in the first two columns of the Young tableau of the gauge field.[12] Then under these two criteria, one can conclude that the only allowed types of gauge fields are represented by rectangular Young tableaux of height $\frac{1}{2}(D-1)$:

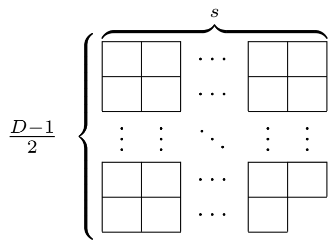


As one can see, the first two columns of h's Young tableau contain already $D-1$ boxes, which is the maximal number allowed by the second criterion mentioned above. Furthermore, the first criterion implies that if we took away one box from a column, we had to put this box back into another column (just like the spin-2 case, this pair of columns should add up to $D-1$ boxes), and thus the total number of boxes in the two longest columns had to exceed the limit $D-1$. This is the reason why there are no allowed types of Young tableaux other than rectangular ones.

Again, one may solve the divergenceless condition in the generalized FP equations using the generalized Einstein tensor, and by substituting the solution into the Klein-Gordon equation and the traceless condition, one obtains the NMG-like equations of motion. The gauge transformation rule for the spin-s gauge field is

[12] Otherwise the corresponding representation of the little group is not valid. See Appendix D for details.

the derivative of a tensor of the type



$$. \qquad (3.2.13)$$

3.3 Factorizing the Klein-Gordon Operator

Just like the 3D cases, the NMG-like models in higher dimensions may also contain ghosts. Take the action (3.2.6) in 7D for instance

$$S_{\text{NMG }\mathcal{Y}_{[3]}} = \int d^7x\ h^{\mu_1\mu_2\mu_3}\left(\Box - m^2\right) G_{\mu_1\mu_2\mu_3}(h)\ . \qquad (3.3.1)$$

In analogy to the analysis at the beginning of Section 2.3.2, one can also do an analysis of the above action. We imposing the gauge-fixing condition

$$\partial^i h_{i\mu\nu} = 0\ , \qquad (3.3.2)$$

under which the gauge field can be decomposed as

$$h_{0ij} = a_{ij} \quad \text{and} \quad h_{ijk} = \varepsilon_{ijk}{}^{lmn}\partial_l b_{mn}\ , \qquad (3.3.3)$$

where $a_{ij} = -a_{ji}$ and $b_{ij} = -b_{ji}$. In consistency with the gauge-fixing, we have $\partial^i a_{ij} = 0$, and we impose $\partial^i b_{ij} = 0$ to eliminate its redundant degrees of freedom.[13] One can count that a and b each carry 10 degrees of freedom. Then by substituting (3.3.3) into (3.3.1) and dropping all terms proportional to the divergence of a or b, we obtain

$$S_{\text{NMG }\mathcal{Y}_{[3]}} = \int d^7x\ 36a^{ij}\left(\Box - m^2\right)\nabla^2 b_{ij}\ . \qquad (3.3.4)$$

By further diagonalizing it one can see among the 20 propagating degrees of freedom, 10 of them have the wrong sign.

Observe that (3.3.1) contains one Levi-Civita tensor, and thus it is a parity odd action, which reminds us of the discussion in Section 2.3.3. We would expect the total 20 propagating degrees of freedom could be equally divided into two parts that have different signs in the action and that are interchanged by a parity transformation. We might expect to construct a TMG-like model with only 10 degrees of freedom that are ghost-free.

[13] One can see that if $b_{mn} = \partial_{[m}\xi_{n]}$, it does not contribute to h.

This is indeed possible. In the following, like what was done in 3D, we will first discuss the generalization of the $\sqrt{\mathrm{FP}}$ theory, then discuss the possibility to boost up the derivatives in order to obtain TMG-like models.

According to a theorem, generically in odd-D dimensions, when breaking the parity symmetry, i.e. going from the group $\mathrm{O}(D-1)$ to $\mathrm{SO}(D-1)$, the irrep represented by a Young tableau that has $\frac{1}{2}(D-1)$ boxes in the first column splits into two non-equivalent irreps represented by the same Young tableau.[14]

To make it explicit, as a generalization of what has been done in Section 2.4.1, in $D = 4k-1$, where $k = 1, 2, \cdots$, for any tensor $T_{\rho_1 \cdots \rho_{2k-1}, \cdots}$ of the type

$$
\begin{array}{|c|c}
\cline{1-1}
\rho_1 & \cdots \\
\cline{1-1}
\rho_2 & \cdots \\
\cline{1-1}
\multicolumn{1}{c}{\vdots} & \ddots \\
\cline{1-1}
\rho_{2k-1} & \\
\cline{1-1}
\end{array} \quad ,
$$

its Klein-Gordon equation $\left(\Box - m^2\right) T_{\rho_1 \cdots \rho_{2k-1}, \cdots} = 0$ under the divergenceless condition can be written as

$$
\begin{aligned}
&\left[\frac{1}{(2k-1)!} \varepsilon_{\mu_1 \cdots \mu_{2k-1}}{}^{\alpha \nu_1 \cdots \nu_{2k-1}} \partial_\alpha \pm m \delta^{\nu_1 \cdots \nu_{2k-1}}_{\mu_1 \cdots \mu_{2k-1}}\right] \\
&\quad \cdot \left[\frac{1}{(2k-1)!} \varepsilon_{\nu_1 \cdots \nu_{2k-1}}{}^{\beta \rho_1 \cdots \rho_{2k-1}} \partial_\beta \mp m \delta^{\rho_1 \cdots \rho_{2k-1}}_{\nu_1 \cdots \nu_{2k-1}}\right] T_{\rho_1 \cdots \rho_{2k-1}, \cdots} = 0 \ , \qquad (3.3.5)
\end{aligned}
$$

where the Klein-Gordon operator has been factorized into two first-order differential operators. Then by dropping the first operator, one obtains a pair of "generalized $\sqrt{\mathrm{FP}}$ equations" of T [15]

$$
\begin{aligned}
\left[\frac{1}{(2k-1)!} \varepsilon_{\nu_1 \cdots \nu_{2k-1}}{}^{\beta \rho_1 \cdots \rho_{2k-1}} \partial_\beta - m \delta^{\rho_1 \cdots \rho_{2k-1}}_{\nu_1 \cdots \nu_{2k-1}}\right] T_{\rho_1 \cdots \rho_{2k-1}, \cdots} &= 0 \ , \\
\left[\frac{1}{(2k-1)!} \varepsilon_{\nu_1 \cdots \nu_{2k-1}}{}^{\beta \rho_1 \cdots \rho_{2k-1}} \partial_\beta + m \delta^{\rho_1 \cdots \rho_{2k-1}}_{\nu_1 \cdots \nu_{2k-1}}\right] T_{\rho_1 \cdots \rho_{2k-1}, \cdots} &= 0 \ ,
\end{aligned}
\qquad (3.3.6)
$$

which are interchanged by a parity transformation and each carry half of the propagating degrees of freedom of the generalized FP equations of T.

As for $D = 4k+1$, one can also write down a similar pair of equations, but unfortunately by taking a product of the two first-order differential operators one obtains $-\left(\Box + m^2\right)$, which is tachyonic, instead of the Klein-Gordon operator.

[14] See e.g. Chapter 10 in [40] for the theorem. A detailed discussion on the spin-1 case has been given in [29].

[15] Note that the divergenceless and traceless conditions can be derived from either of the two equations.

Therefore, in this situation we do not have a direct generalization of the $\sqrt{\mathrm{FP}}$ equations.

We now only focus on the $D = 4k - 1$ situation, and we would like to solve the divergenceless condition to boost up the derivatives, in the hope that (3.3.6) can be converted into a TMG-like model. As already discussed in the previous section, in order to achieve this, we can only use a gauge field whose generalized Einstein tensor, defined as the dual of its generalized Riemann tensor, lives in the same representation as the gauge field itself. Then it is obvious that for spin-s ($s \geq 1$) the only allowed type of gauge field is represented by a rectangular Young tableau of width s and height $2k - 1$. After substituting the solution to the divergenceless condition $T = G(h)$ into (3.3.6) one obtains a TMG-like model with a gauge symmetry parameterized by a tensor of the type (3.2.13). More technical details will be illustrated by the 7D spin-2 example in the next section.

3.4 An Example in 7D

As already explained, for any massive spin $s \geq 1$, the only type of gauge field that is suitable for both NMG and TMG-like models is represented by a rectangular Young tableau of height $2k - 1$ in $4k - 1$ dimensions. In this section, we would like to discuss the specific example of $s = 2$ and $k = 2$, i.e. the type $\begin{array}{|c|c|}\hline \, & \, \\ \hline \, & \, \\ \hline \, & \, \\ \hline\end{array}$ in 7D, in order to illustrate some technical details. We will see that the 7D models are in many aspects similar to the 3D NMG and TMG at the linearized level, and in the last subsection we will discuss the issue of going beyond the linearized level [41].

To simplify the notation, we use the index with a bar to denote a set of three antisymmetrized indices, e.g. $\bar{\mu}$ stands for the set of indices $\mu_1\mu_2\mu_3$ that are antisymmetrized.

3.4.1 The Models

We start from the tensor field $T_{\bar{\mu},\bar{\nu}}$ of the type $\begin{array}{|c|c|}\hline \, & \, \\ \hline \, & \, \\ \hline \, & \, \\ \hline\end{array}$. The representation $\begin{array}{|c|c|}\hline \, & \, \\ \hline \, & \, \\ \hline \, & \, \\ \hline\end{array}$ of the little group SO(6) carries 70 degrees of freedom, so let us first check that after the Lorentz covariantizaton, with the divergenceless and traceless conditions imposed, $T_{\bar{\mu},\bar{\nu}}$ carries the same number of degrees of freedom. In the following counting, Young tableaux should be thought as representations of GL(7).

Without any constraints, the number of degrees of freedom carried by $T_{\bar{\mu},\bar{\nu}}$ is

$$\begin{array}{|c|c|}\hline \, & \, \\ \hline \, & \, \\ \hline \, & \, \\ \hline\end{array} = 490 \ . \tag{3.4.1}$$

Now we impose the divergenceless condition $\partial^{\mu_1} T_{\bar{\mu},\bar{\nu}} = 0$.[16] At first sight, this constraint appears to cut the number of degrees of freedom of $\ydiagram{2,2,1}$, but in fact this corresponds to an excessive cutting, because the double-divergence $\partial^{\mu_1}\partial^{\mu_2} T_{\bar{\mu},\bar{\nu}}$ vanishes by construction, which means the degrees of freedom represented by $\ydiagram{2,1,1}$ has to be compensated. However, this compensation is again an excessive compensation, because the triple-divergence $\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3} T_{\bar{\mu},\bar{\nu}}$ vanishes by construction, and we must re-cut the degrees of freedom of $\ydiagram{1,1,1}$. Therefore, in total

$$\ydiagram{2,2,1} - \ydiagram{2,1,1} + \ydiagram{1,1,1} = 490 - 210 + 35 = 315 \tag{3.4.2}$$

degrees of freedom are suppressed by the divergenceless condition.

Then we impose the traceless condition. This means the number of independent components carried by every Young tableau should be reduced by removing its trace. Therefore, the number of degrees of freedom taken away by the traceless condition should be equal to

$$\ydiagram{2,2} - \ydiagram{2,1} + \ydiagram{1,1} = 196 - 112 + 21 = 105 \ , \tag{3.4.3}$$

where the first Young tableau stands for the trace of $T_{\bar{\mu},\bar{\nu}}$, and the second (third) stands for the trace of the divergence (double-divergence) of $T_{\bar{\mu},\bar{\nu}}$.

In the end, we find that $T_{\bar{\mu},\bar{\nu}}$ under both constraints indeed carries 70 degrees of freedom:

$$490 - 315 - 105 = 70 \ . \tag{3.4.4}$$

Then we can write down the generalized FP equations:

$$\left(\Box - m^2\right) T_{\bar{\mu},\bar{\nu}} = 0 \quad , \quad \partial^{\mu_1} T_{\bar{\mu},\bar{\nu}} = 0 \quad , \quad \eta^{\mu_1\nu_1} T_{\bar{\mu},\bar{\nu}} = 0 \ , \tag{3.4.5}$$

which describes 70 propagating degrees of freedom, and the generalized $\sqrt{\mathrm{FP}}$ equation:

$$\left(\frac{1}{6}\varepsilon_{\bar{\mu}}{}^{\alpha\bar{\rho}}\partial_\alpha - \mu\delta_{\bar{\mu}}^{\bar{\rho}}\right) T_{\bar{\rho},\bar{\nu}} = 0 \ , \tag{3.4.6}$$

where $\mu = \pm m$, and each sign corresponds to an equation describing 35 propagating degrees of freedom. Note that the divergenceless and traceless condtions can be derived from (3.4.6).

In the next step, one can solve the divergenceless condition by

$$T_{\bar{\mu},\bar{\nu}} = G_{\bar{\mu},\bar{\nu}}\left(h\right) \equiv \varepsilon_{\bar{\mu}}{}^{\alpha\bar{\rho}}\varepsilon_{\bar{\nu}}{}^{\beta\bar{\sigma}}\partial_\alpha\partial_\beta h_{\bar{\rho},\bar{\sigma}} \ , \tag{3.4.7}$$

where the gauge field h and the generalized Einstein tensor $G\left(h\right)$ also belong to the type $\ydiagram{2,2,1}$. Thus one may boost up the derivatives to derive the NMG-like equations of motion

$$\left(\Box - m^2\right) G_{\bar{\mu},\bar{\nu}}\left(h\right) = 0 \quad , \quad \eta^{\mu_1\nu_1} G_{\bar{\mu},\bar{\nu}}\left(h\right) = 0 \ , \tag{3.4.8}$$

[16]Because T satisfies $T_{\bar{\mu},\bar{\nu}} = T_{\bar{\nu},\bar{\mu}}$, the divergenceless condition $\partial^{\mu_1} T_{\bar{\mu},\bar{\nu}} = 0$ is equivalent to $\partial^{\nu_1} T_{\bar{\mu},\bar{\nu}} = 0$.

and the TMG-like equation of motion

$$\left(\frac{1}{6}\varepsilon_{\bar{\mu}}{}^{\alpha\bar{\rho}}\partial_\alpha - \mu\delta^{\bar{\rho}}_{\bar{\mu}}\right) G_{\bar{\rho},\bar{\nu}}(h) = 0 \ , \tag{3.4.9}$$

from which one can also derive $\eta^{\mu_1\nu_1}G_{\bar{\mu},\bar{\nu}}(h) = 0$.

Analogous to 3D, we can define the generalized Cotton tensor[17]

$$C_{\bar{\mu},\bar{\nu}}(h) = \mathcal{Y}_{[3,3]}\left[\varepsilon_{\bar{\mu}}{}^{\alpha\bar{\rho}}\partial_\alpha G_{\bar{\rho},\bar{\nu}}(h)\right] \ , \tag{3.4.10}$$

which is both divergenceless and traceless, and we can use it to construct both the NMG-like and the TMG-like actions without auxiliary fields.

The NMG-like action reads

$$S_{\text{7D NMG spin-2}} = \int d^7x \left\{\frac{1}{72}h^{\bar{\mu},\bar{\nu}}\varepsilon_{\bar{\mu}}{}^{\alpha\bar{\rho}}\partial_\alpha C_{\bar{\rho},\bar{\nu}}(h) - \frac{1}{2}m^2 h^{\bar{\mu},\bar{\nu}}G_{\bar{\mu},\bar{\nu}}(h)\right\} \ , \tag{3.4.11}$$

from which one can derive[18]

$$\frac{1}{36}\varepsilon_{\bar{\mu}}{}^{\alpha\bar{\rho}}\partial_\alpha C_{\bar{\rho},\bar{\nu}}(h) - m^2 G_{\bar{\mu},\bar{\nu}}(h) = 0 \ . \tag{3.4.12}$$

By taking the trace of this equation one obtains

$$\eta^{\mu_1\nu_1}G_{\bar{\mu},\bar{\nu}}(h) = 0 \ , \tag{3.4.13}$$

and substituting it back into (3.4.12) gives

$$\left(\Box - m^2\right) G_{\bar{\mu},\bar{\nu}}(h) = 0 \ . \tag{3.4.14}$$

Thus (3.4.8) is derived.

The TMG-like action reads

$$S_{\text{7D TMG spin-2}} = \int d^7x \left\{\frac{1}{12}h^{\bar{\mu},\bar{\nu}}C_{\bar{\mu},\bar{\nu}}(h) - \frac{1}{2}\mu h^{\bar{\mu},\bar{\nu}}G_{\bar{\mu},\bar{\nu}}(h)\right\} \ , \tag{3.4.15}$$

from which one can derive

$$\frac{1}{6}C_{\bar{\mu},\bar{\nu}}(h) - \mu G_{\bar{\mu},\bar{\nu}}(h) = 0 \ , \tag{3.4.16}$$

and the trace of this equation gives the traceless condition

$$\eta^{\mu_1\nu_1}G_{\bar{\mu},\bar{\nu}}(h) = 0 \ . \tag{3.4.17}$$

Using this condition one can prove that (3.4.16) and (3.4.9) are equivalent.

[17] Note that off-shell we need the Young symmetrizer, otherwise $C_{\bar{\mu},\bar{\nu}}(h)$ does not satisfy the symmetry $\tiny\begin{array}{|c|c|}\hline \, & \, \\ \hline \, & \, \\ \hline \, & \, \\ \hline\end{array}$. However, on-shell by using the traceless condition of $G_{\bar{\mu},\bar{\nu}}(h)$ one can prove that dropping the Young symmetrizer gives an equivalent formula.

[18] Because the generalized Cotton tensor is both divergenceless and traceless, one can prove that the first term in this equation of motion, without being projected by a Young symmetrizer, already satisfies the symmetry $\tiny\begin{array}{|c|c|}\hline \, & \, \\ \hline \, & \, \\ \hline \, & \, \\ \hline\end{array}$.

3.4.2 Canonical Analysis

We will verify, by canonical analysis, that the actions (3.4.11) and (3.4.15) indeed describe 70 and 35 (for each choice of μ) propagating degrees of freedom, respectively, which are ghost-free.

Before the analysis, we first count the number of degrees of freedom carried by the gauge field h. The gauge field h of the type $\ydiagram{2,2,2}$, which is counted as the **490** representation of GL(7), transforms under the gauge transformations

$$\delta h_{\mu_1\mu_2\mu_3,\nu_1\nu_2\nu_3} = \mathcal{Y}_{[3,3]}\left(\partial_{[\nu_3|}\xi_{\mu_1\mu_2\mu_3,|\nu_1\nu_2]}\right) \ , \tag{3.4.18}$$

where ξ is of the type $\ydiagram{2,2,1}$, which has 490 degrees of freedom. We should be careful with the counting because ξ has its own gauge symmetry

$$\delta\xi_{\mu_1\mu_2\mu_3,\nu_1\nu_2} = \mathcal{Y}_{[3,2]}\left(\partial_{[\nu_2|}\zeta_{\mu_1\mu_2\mu_3,|\nu_1]}\right) \ , \tag{3.4.19}$$

where ζ is of the type $\ydiagram{2,1,1}$ with 210 degrees of freedom. Moreover ζ has also by itself a gauge symmetry

$$\delta\zeta_{\mu_1\mu_2\mu_3,\nu_1} = \mathcal{Y}_{[3,1]}\left(\partial_{\nu_1}\lambda_{\mu_1\mu_2\mu_3}\right) \ , \tag{3.4.20}$$

where λ is an antisymmetric tensor and has 35 degrees of freedom. Therefore, to summarize the above counting, we expect that after a gauge-fixing h should carry

$$490 - 490 + 210 - 35 = 175 \tag{3.4.21}$$

degrees of freedom.

Similarly to Section 2.3, we split the indices into temporal and spatial components like $\mu = (0, i)$, $i = 1, \cdots, 6$, and impose the gauge-fixing condition

$$\partial^i h_{i\mu_2\mu_3,\nu_1\nu_2\nu_3} = 0 \ . \tag{3.4.22}$$

To see that the gauge degrees of freedom cannot propagate under this condition, we may vary (3.4.22) with respect to (3.4.18) and require this variation to be zero, which leads to a condition on ξ with a gauge symmetry that can be fixed by imposing the following restriction:

$$\partial^{i_2}\xi_{i_2\mu_3,\nu_1\nu_2\nu_3} = 0 \ . \tag{3.4.23}$$

Then we obtain

$$\nabla^2\xi_{\mu_2\mu_3,\nu_1\nu_2\nu_3} = 0 \ , \tag{3.4.24}$$

which means ξ cannot propagate. Moreover, in a similar manner, in order to see ζ and λ cannot carry any propagating degrees of freedom, we can require the variation of (3.4.23) with respect to (3.4.19) to be zero, while gauge-fixing

$$\partial^{i_3}\zeta_{i_3,\nu_1\nu_2\nu_3} = 0 \ , \tag{3.4.25}$$

and also require the variation of (3.4.25) with respect to (3.4.20) to be zero.

Under the gauge-fixing condition (3.4.22), we parameterize h in terms of the independent components (a, b, c, d, e) as follows:[19]

$$
\begin{aligned}
h_{0i_2i_3,0j_2j_3} &= a_{i_2i_3,j_2j_3} \ , \\
h_{0i_2i_3,j_1j_2j_3} &= \varepsilon_{j_1j_2j_3}{}^{k_1k_2k_3}\partial_{k_1} b_{k_2k_3,i_2i_3} + \Bigg\{ \left(\delta_{i_3j_3} - \frac{\partial_{i_3}\partial_{j_3}}{\nabla^2} \right) c_{j_1j_2,i_2} \\
&\quad + \left(\delta_{i_2j_2}\delta_{i_3j_3} - \frac{\partial_{i_2}\partial_{j_2}}{\nabla^2}\delta_{i_3j_3} - \delta_{i_2j_2}\frac{\partial_{i_3}\partial_{j_3}}{\nabla^2} \right) d_{j_1} \Bigg\}_{\text{a.s.}} \ , \\
h_{i_1i_2i_3,j_1j_2j_3} &= \varepsilon_{i_1i_2i_3}{}^{k_1k_2k_3}\varepsilon_{j_1j_2j_3}{}^{l_1l_2l_3}\partial_{k_1}\partial_{l_1} e_{k_2k_3,l_2l_3} \ ,
\end{aligned} \tag{3.4.26}
$$

where the properties of these components are listed below.

Components	Symmetry	divergence	trace	degrees
$a_{..,..}$	$\ydiagram{2,2}$	-less	-ful	50
$b_{..,..}$	$\ydiagram{2,2}$	-less	-less	35
$c_{..,.}$	$\ydiagram{2,1}$	-less	-less	35
$d_{.}$	$\square$	-less		5
$e_{..,..}$	$\ydiagram{2,2}$	-less	-ful	50

Table 3.1: Properties of the components of h

As shown in the last column of this table, the total number of degrees of freedom sums up to 175, which is consistent with (3.4.21).

In the next step, we substitute (3.4.26) into each term of (3.4.11) and (3.4.15). Then we separate the trace of $a_{i_2i_3,j_2j_3}$ from its traceless part:

$$
\begin{aligned}
a_{i_2i_3,j_2j_3} &= \hat{a}_{i_2i_3,j_2j_3} + \Bigg\{ \left(\eta_{i_2j_2} - \frac{\partial_{i_2}\partial_{j_2}}{\nabla^2} \right) \bar{a}_{i_3,j_3} \\
&\quad + \left(\eta_{i_2j_2}\eta_{i_3j_3} - \frac{\partial_{i_2}\partial_{j_2}}{\nabla^2}\eta_{i_3j_3} - \eta_{i_2j_2}\frac{\partial_{i_3}\partial_{j_3}}{\nabla^2} \right) a \Bigg\}_{\text{a.s.}} \ ,
\end{aligned} \tag{3.4.27}
$$

where $\hat{a}_{i_2i_3,j_2j_3}$ and $\bar{a}_{i_3,j_3}$ are traceless and carry 35 and 14 degrees of freedom, which represent the traceless and the trace parts of $a_{i_2i_3,j_2j_3}$, respectively, and a carrying one degree of freedom represents the double trace part. We also split

[19] The notation $\{\ \}_{\text{a.s.}}$ stands for antisymmetrizing all indices within the curly bracket that have the same Latin letter. For instance, $\{T_{i_2i_3j_1j_2j_3}\}_{\text{a.s.}} = T_{[i_2i_3][j_1j_2j_3]}$.

$e_{i_2 i_3, j_2 j_3}$ in the same way. The resulting Lagrangian terms are

$$\begin{aligned} h^{\bar{\mu},\bar{\nu}} G_{\bar{\mu},\bar{\nu}}(h) &= 2\,(3!)^4\, \hat{a}^{i_2 i_3, j_2 j_3} \left(\nabla^2\right)^2 \hat{e}_{i_2 i_3, j_2 j_3} \\ &+2\,(3!)^4\, b^{i_2 i_3, j_2 j_3} \left(\nabla^2\right)^2 b_{i_2 i_3, j_2 j_3} \\ &+\frac{3}{2}\,(3!)^4\, \bar{a}^{i_3, j_3} \left(\nabla^2\right)^2 \bar{e}_{i_3, j_3} + 20\,(3!)^4\, a \left(\nabla^2\right)^2 e \\ &+\frac{3}{5}\,(5!)\, c^{j_1 j_2, i_2} \nabla^2 c_{j_1 j_2, i_2} - 9\,(2!4!)\, d^{j_1} \nabla^2 d_{j_1} \ , \end{aligned} \qquad (3.4.28)$$

$$h^{\bar{\mu},\bar{\nu}} C_{\bar{\mu},\bar{\nu}}(h) = -6\,(3!)^4\, b^{i_2 i_3, j_2 j_3} \left(\nabla^2\right)^2 \left(\hat{a}_{i_2 i_3, j_2 j_3} + 4\Box \hat{e}_{i_2 i_3, j_2 j_3}\right) \ , \qquad (3.4.29)$$

$$\begin{aligned} h^{\bar{\mu},\bar{\nu}} \varepsilon_{\bar{\mu}}{}^{\alpha\bar{\rho}} \partial_\alpha C_{\bar{\rho},\bar{\nu}}(h) &= 2\,(3!)^6\, b^{i_2 i_3, j_2 j_3} \left(\nabla^2\right)^2 \Box b_{i_2 i_3, j_2 j_3} \\ &+ (3!)^6\, \hat{a}^{i_2 i_3, j_2 j_3} \left(\nabla^2\right)^2 \Box \hat{e}_{i_2 i_3, j_2 j_3} \\ &+2\,(3!)^6\, \hat{e}^{i_2 i_3, j_2 j_3} \left(\nabla^2\right)^2 \Box^2 \hat{e}_{i_2 i_3, j_2 j_3} \\ &+\frac{1}{8}\,(3!)^6\, \hat{a}^{i_2 i_3, j_2 j_3} \left(\nabla^2\right)^2 \hat{a}_{i_2 i_3, j_2 j_3} \ . \end{aligned} \qquad (3.4.30)$$

By substituting (3.4.30) and (3.4.28) into the NMG-like action (3.4.11), and doing the field redefinition:

$$\hat{a}_{i_2 i_3, j_2 j_3} = \tilde{a}_{i_2 i_3, j_2 j_3} - 4\Box \hat{e}_{i_2 i_3, j_2 j_3} + 8m^2 \hat{e}_{i_2 i_3, j_2 j_3} \ , \qquad (3.4.31)$$

we obtain

$$\begin{aligned} S_{\text{7D NMG spin-2}} = \int d^7x \Big\{ & (3!)^4\, b^{i_2 i_3, j_2 j_3} \left(\nabla^2\right)^2 \left(\Box - m^2\right) b_{i_2 i_3, j_2 j_3} \\ &+4\,(3!)^4\, m^2 \hat{e}^{i_2 i_3, j_2 j_3} \left(\nabla^2\right)^2 \left(\Box - m^2\right) \hat{e}_{i_2 i_3, j_2 j_3} \\ &+\frac{1}{16}\,(3!)^4\, \tilde{a}^{i_2 i_3, j_2 j_3} \left(\nabla^2\right)^2 \tilde{a}_{i_2 i_3, j_2 j_3} \\ &-\frac{3}{4}\,(3!)^4\, m^2 \bar{a}^{i_3, j_3} \left(\nabla^2\right)^2 \bar{e}_{i_3, j_3} - 10\,(3!)^4\, m^2 a \left(\nabla^2\right)^2 e \\ &-\frac{3}{10}\,(5!)\, m^2 c^{j_1 j_2, i_2} \nabla^2 c_{j_1 j_2, i_2} + \frac{9}{2}\,(2!4!)\, m^2 d^{j_1} \nabla^2 d_{j_1} \Big\} \ . \end{aligned} \qquad (3.4.32)$$

We see that the first and second terms each represent 35 propagating degrees of freedom, both with the right sign, and the rest part decouples and does not propagate. Thus, this action describes 70 massive physical degrees of freedom.

By substituting (3.4.29) and (3.4.28) into the TMG-like action (3.4.15), and doing the field redefinitions:

$$\hat{a}_{i_2 i_3, j_2 j_3} = \tilde{a}_{i_2 i_3, j_2 j_3} - \frac{2}{\mu} \Box b_{i_2 i_3, j_2 j_3} \quad , \quad \hat{e}_{i_2 i_3, j_2 j_3} = \tilde{e}_{i_2 i_3, j_2 j_3} - \frac{1}{2\mu} b_{i_2 i_3, j_2 j_3} \ , \qquad (3.4.33)$$

we obtain

$$\begin{aligned} S_{\text{7D TMG spin-2}} \quad = \quad & \int d^7x \left\{ \frac{1}{\mu} (3!)^4 \, b^{i_2 i_3, j_2 j_3} \left(\nabla^2\right)^2 \left(\Box - \mu^2\right) b_{i_2 i_3, j_2 j_3} \right. \\ & - (3!)^4 \, \mu \tilde{a}^{i_2 i_3, j_2 j_3} \left(\nabla^2\right)^2 \tilde{e}_{i_2 i_3, j_2 j_3} \\ & - \frac{3}{4} (3!)^4 \, \mu \bar{a}^{i_3, j_3} \left(\nabla^2\right)^2 \bar{B}_{i_3, j_3} - 10 \, (3!)^4 \, \mu a \left(\nabla^2\right)^2 e \\ & \left. - \frac{3}{10} (5!) \, \mu c^{j_1 j_2, i_2} \nabla^2 c_{j_1 j_2, i_2} + \frac{9}{2} (2!4!) \, \mu d^{j_1} \nabla^2 d_{j_1} \right\} . \end{aligned} \tag{3.4.34}$$

The first term contains 35 propagating degrees of freedom and the rest terms are all auxiliary, and by properly choosing an overall sign, this action is ghost-free.

3.4.3 Discussion

As can be seen, the above 7D models are very similar to the 3D linearized NMG and TMG models. The 7D actions (3.4.11) and (3.4.15) look almost the same as the 3D actions (2.2.15) and (2.4.11), except for bars on the indices and some coefficients. The canonical analysis is also similar.

There are even more similarities. For instance, it is interesting to consider the $m \to 0$ limit of the 7D models. Using (3.4.30) one can prove that the massless limit of the 7D NMG-like model carries 35 massless propagating degrees of freedom, which is half of the degrees of freedom of the corresponding massive model. This is similar to the 3D situation, where, according to a similar analysis in [28], the massless limit of the linearized NMG contains one massless propagating degree of freedom. Furthermore, using (3.4.29) one can prove that the massless limit of 7D TMG-like model has no propagating degree of freedom, which is the same as taking the limit of 3D linearized TMG.

The crucial question remains whether the 7D extensions we discussed are curiosities of the linearized approximation or whether one can go beyond the linearized approximation and introduce non-trivial interactions. This is a non-trivial issue in view of the fact that we are using non-standard representations to describe the massive "spin-2" particle. Perhaps, a slightly easier question to ask is whether one can introduce interactions for only the mass term, i.e. the term with two derivatives. For both the NMG-like and the TMG-like actions this term is given by

$$S\left[h\right] = \int d^7x \left\{ \frac{1}{2} h^{\bar{\mu},\bar{\nu}} G_{\bar{\mu},\bar{\nu}}\left(h\right) \right\} \, . \tag{3.4.35}$$

This term by itself leads to the equation of motion $G(h) = 0$ and therefore does not describe any degree of freedom, as one would expect from a mass term. Given that there are no propagating degrees of freedom one might hope that it will be an easier task to construct interactions.

The model (3.4.35) is the 7D version of the 3D gravity action that neither describes any degree of freedom. The 3D gravity action has the interesting feature that it can be reformulated as a Chern-Simons (CS) action [42, 43]. In order to achieve this, one must use a first-order formalism with the Dreibein $e_\mu{}^a$ and spin-connection $\omega_\mu{}^a$ as independent fields. Writing $e_\mu{}^a = \delta_\mu{}^a + h_\mu{}^a$ this 3D CS action is at the linearized level given by

$$S_{\rm CS}\,[h,\omega] = \int d^3x\,\varepsilon^{\mu\nu\rho}\left\{\omega_\mu{}^a\partial_\nu h_\rho{}^b\eta_{ab} - \frac{1}{2}\omega_\mu{}^a\delta_\nu{}^b\omega_\rho{}^c\varepsilon_{abc}\right\}\,. \tag{3.4.36}$$

It is invariant under the linearized Lorentz transformation

$$\delta h_{\mu a} = \Lambda_{\mu a} \quad , \quad \delta\omega_\mu{}^a = -\frac{1}{2}\varepsilon^{abc}\partial_\mu\Lambda_{bc}\ , \tag{3.4.37}$$

where $\Lambda_{\mu a} = -\Lambda_{a\mu}$. These linearized gauge transformations can be fixed by imposing the gauge-fixing condition $h_{\mu a} = h_{a\mu}$. One then obtains a first-order action in terms of $\omega_\mu{}^a$ and a symmetric tensor $h_{\mu\nu}$. One of the reasons that this action can be extended to include interactions is that the Kronecker delta $\delta_\alpha{}^b$, occurring in the action (3.4.36), is in the same representation as the Dreibein $e_\mu{}^a$ and, therefore, can become part of this Dreibein at the non-linear level. The interactions are then determined by introducing the non-abelian CS structure, dictated by the Lorentz structure of the different gauge fields.

It turns out that a similar first-order formulation exists of the model defined by the action (3.4.35) in terms of two fields $h_{\bar\mu,\bar\nu}$ and $\omega_{\bar\mu,\bar\nu}$ which both have the symmetry properties corresponding to the Young tableau

$$\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}\,. \tag{3.4.38}$$

Similar to [44], at the quadratic level such a first-order action can be written in the following form

$$S[h,\omega] = \int d^7x\ \varepsilon^{\bar\mu\alpha\bar\nu}\left\{\omega_{\bar\mu,}{}^{\bar\rho}\partial_\alpha h_{\bar\nu,\bar\rho} - \frac{1}{72}\omega_{\bar\mu,}{}^{\bar\sigma}\delta_\alpha{}^\beta\omega_{\bar\nu,}{}^{\bar\tau}\varepsilon_{\bar\sigma\beta\bar\tau}\right\}\ . \tag{3.4.39}$$

This action has a gauge invariance under a "generalized" linearized Lorentz transformation, with parameters $\Lambda_{\mu_1\mu_2,\nu_1\nu_2\nu_3\nu_4}$, given by

$$\begin{aligned}
\delta h_{\bar\mu,\bar\nu} &= \Lambda_{[\mu_1\mu_2,\mu_3]\nu_1\nu_2\nu_3}\ ,\\
\delta\omega_{\bar\rho,}{}^{\bar\mu} &= \varepsilon^{\bar\mu\alpha\bar\nu}\partial_\alpha\Lambda_{\nu_1\nu_2,\nu_3\rho_1\rho_2\rho_3} - \frac{1}{4}\delta^{\bar\mu}_{\bar\rho}\varepsilon^{\bar\sigma\alpha\bar\nu}\partial_\alpha\Lambda_{\nu_1\nu_2,\nu_3\sigma_1\sigma_2\sigma_3}\\
&\quad + \Big\{-\frac{9}{2}\delta^{\mu_1}_{\rho_1}\varepsilon^{\sigma_1\mu_2\mu_3\alpha\bar\nu}\partial_\alpha\Lambda_{\nu_1\nu_2,\nu_3\sigma_1\rho_2\rho_3}\\
&\qquad + 3\delta^{\mu_1\mu_2}_{\rho_1\rho_2}\varepsilon^{\sigma_1\sigma_2\mu_3\alpha\bar\nu}\partial_\alpha\Lambda_{\nu_1\nu_2,\nu_3\sigma_1\sigma_2\rho_3}\Big\}_{\rm a.s.}\ .
\end{aligned} \tag{3.4.40}$$

In effect, the Λ-transformation represents three independent gauge transformations parameterized by fields corresponding to three irreps:

$$\ydiagram{1,1} \otimes \ydiagram{1,1,1,1} \to \ydiagram{1,1,1,1,1,1} \oplus \ydiagram{2,1,1,1,1} \oplus \ydiagram{2,2,1,1} \,. \tag{3.4.41}$$

The gauge transformations (3.4.40) are the generalization of the 3D Lorentz transformations (3.4.37).

It is easy to see that the action (3.4.39) is equivalent to (3.4.35). One first imposes the condition

$$h_{\bar{\mu},\bar{\nu}} = \mathcal{Y}_{[3,3]}\, h_{\bar{\mu},\bar{\nu}} \tag{3.4.42}$$

to fix the gauge transformations (3.4.40). Next, one uses the equation of motion for $\omega_{\bar{\mu},\bar{\nu}}$ to solve for $\omega_{\bar{\mu},\bar{\nu}}$ in terms of $h_{\bar{\mu},\bar{\nu}}$:

$$\omega_{\bar{\mu},\bar{\nu}} = \epsilon_{\bar{\nu}}{}^{\alpha\bar{\rho}} \partial_\alpha h_{\bar{\rho},\bar{\mu}} \,. \tag{3.4.43}$$

Note that this equation implies that $\omega_{\bar{\mu},\bar{\nu}}$ is traceless, i.e. $\eta^{\mu_1\nu_1}\omega_{\bar{\mu},\bar{\nu}} = 0$. Substituting this solution back into (3.4.39) the two terms in (3.4.39) coincide and become identical to the single term in (3.4.35).

The gauge-invariant first-order formulation we have obtained at this point resembles the 3D CS structure. There are, however, also important differences. First of all, it is not clear how to introduce in the 7D case the notion of flat and curved indices thereby to anticipate a possible CS-like structure. A related issue is that we are working now with tensors instead of gauge vectors. It is not obvious how to introduce non-abelian structures for these tensors. The structure we have obtained so far suggests an extension of CS terms for vectors to a "generalized CS" structure for a non-abelian version of free differential algebras. An alternative approach to introduce interactions could be to use a bi-metric formulation. One metric describes the massive spin-2 particle and is used to absorb the $h_{\bar{\mu},\bar{\nu}}$ field, while the other metric is a reference metric that can be used to absorb the Kronecker delta that occurs in the second term of (3.4.39). For now, we leave these possibilities as intriguing open issues.

Since the models in 3D and 7D are so similar, we would expect this is a generic situation for $D = 4k-1$, $k = 1, 2, \cdots$. To finish this section, we give the NMG and TMG-like spin-2 actions for fields of rectangular types of Young tableaux in $4k-1$ dimensions, without further discussion. In the following, we denote the index with a bar as a set of $2k-1$ antisymmetrized indices.

Using the gauge field $h_{\bar{\mu},\bar{\nu}} = \mathcal{Y}_{[2k-1,2k-1]} h_{\bar{\mu},\bar{\nu}}$, the NMG-like action reads

$$S_{\text{NMG}\ \mathcal{Y}_{[2k-1,2k-1]}} = \int d^{4k-1}x \left\{ \frac{1}{2\left[(2k-1)!\right]^2} h^{\bar{\mu},\bar{\nu}} \varepsilon_{\bar{\mu}}{}^{\alpha\bar{\rho}} \partial_\alpha C_{\bar{\rho},\bar{\nu}}\left(h\right) - \frac{1}{2} m^2 h^{\bar{\mu},\bar{\nu}} G_{\bar{\mu},\bar{\nu}}\left(h\right) \right\} \ , \tag{3.4.44}$$

and the TMG-like action reads

$$S_{\text{TMG}\ \mathcal{Y}_{[2k-1,2k-1]}} = \int d^{4k-1}x \left\{ \frac{1}{2\left(2k-1\right)!} T^{\bar{\mu},\bar{\nu}} C_{\bar{\mu},\bar{\nu}}\left(h\right) - \frac{1}{2} \mu T^{\bar{\mu},\bar{\nu}} G_{\bar{\mu},\bar{\nu}}\left(h\right) \right\} \ , \tag{3.4.45}$$

where

$$G_{\bar{\mu},\bar{\nu}}\left(h\right) = \varepsilon_{\bar{\mu}}{}^{\alpha\bar{\rho}} \varepsilon_{\bar{\nu}}{}^{\beta\bar{\sigma}} \partial_\alpha \partial_\beta h_{\bar{\rho},\bar{\sigma}} \ , \tag{3.4.46}$$

and

$$C_{\bar{\mu},\bar{\nu}}\left(h\right) = \mathcal{Y}_{[2k-1,2k-1]} \left[\varepsilon_{\bar{\mu}}{}^{\alpha\bar{\rho}} \partial_\alpha G_{\bar{\rho},\bar{\nu}}\left(h\right) \right] \ . \tag{3.4.47}$$

Chapter 4

On 3D Fermions and Supergravity

4.1 NMG/TMG-Like Models for Fermions

In this section, we will extend the NMG-like and the TMG-like models to fermions (linearized on the flat background). We will first introduce the lower-derivative models, then we will see that the "boosting up derivatives" procedure also applies to fermions in 3D. Actions for gravitini will be constructed.

4.1.1 The Lower-Derivative Model

In 3D we may write down a set of equations for a massive fermion of spin-s in analogy to the bosonic FP equations:

$$\left(\Box - m^2\right)\zeta_{\mu_1\mu_2\cdots\mu_{s-\frac{1}{2}}} = 0 \,, \tag{4.1.1a}$$

$$\partial^{\mu_1}\zeta_{\mu_1\mu_2\cdots\mu_{s-\frac{1}{2}}} = 0 \,, \tag{4.1.1b}$$

$$\gamma^{\mu_1}\zeta_{\mu_1\mu_2\cdots\mu_{s-\frac{1}{2}}} = 0 \,. \tag{4.1.1c}$$

We use a tensor-spinor $\zeta_{(\mu_1\mu_2\cdots\mu_{s-\frac{1}{2}})}$ with $s-\frac{1}{2}$ symmetrized spacetime indices, which is under the constraints of the divergenceless condition (4.1.1b) and the γ-traceless condition (4.1.1c). Suppose ζ is Majorana,[1] one may check the counting that under the two constraints it contains 2 degrees of freedom. Furthermore, we use the Klein-Gordon equation (4.1.1a) to describe its free propagation.

There also exists a formalism in analogy to the bosonic $\sqrt{\text{FP}}$ equations. Since the Klein-Gordon operator can be factorized as

$$\left(\Box - m^2\right) = \left(\partial\!\!\!/ \pm m\right)\left(\partial\!\!\!/ \mp m\right) \,, \tag{4.1.2}$$

[1] The choice of Majorana fermions is analogous to the choice of real fields in the bosonic case. In 3D, a Majorana spinor has two components, which are not necessarily real, but under the Majorana condition the spinor carries only two degrees of freedom. In this chapter, all fermions are Majorana, unless otherwise specified.

we can write down in 3D a pair of Dirac equations

$$\begin{aligned}\left(\not\partial - m\right)\zeta_{\mu_1\mu_2\cdots\mu_{s-\frac{1}{2}}} &= 0\ ,\\ \left(\not\partial + m\right)\zeta_{\mu_1\mu_2\cdots\mu_{s-\frac{1}{2}}} &= 0\ ,\end{aligned} \tag{4.1.3}$$

which are interchanged under the parity transformation.[2] Thus, keeping the two constraints unchanged, we can split the two propagating degrees of freedom into two sets of $\sqrt{\text{FP}}$-like equations[3]:

$$\left(\not\partial - \mu\right)\zeta_{\mu_1\mu_2\cdots\mu_{s-\frac{1}{2}}} = 0\ , \tag{4.1.4a}$$

$$\partial^{\mu_1}\zeta_{\mu_1\mu_2\cdots\mu_{s-\frac{1}{2}}} = 0\ , \tag{4.1.4b}$$

$$\gamma^{\mu_1}\zeta_{\mu_1\mu_2\cdots\mu_{s-\frac{1}{2}}} = 0\ , \tag{4.1.4c}$$

where $\mu = \pm m$ corresponding to two opposite helicities.

4.1.2 Boosting Up the Derivatives

In the bosonic theory, we always solve the divergenceless condition using the generalized Einstein tensor, and substitute the solution into the other equations, which gives higher-derivative models with gauge symmetry. In the fermionic case, we can do exactly the same thing. We can solve $\partial^{\mu_1}\zeta_{\mu_1\mu_2\cdots\mu_{s-\frac{1}{2}}} = 0$ by

$$\zeta_{\mu_1\mu_2\cdots\mu_{s-\frac{1}{2}}} = \mathcal{G}_{\mu_1\mu_2\cdots\mu_{s-\frac{1}{2}}}(\rho) \equiv \varepsilon_{\mu_1}{}^{\nu_1\rho_1}\cdots\varepsilon_{\mu_{s-\frac{1}{2}}}{}^{\nu_{s-\frac{1}{2}}\rho_{s-\frac{1}{2}}}\partial_{\nu_1}\cdots\partial_{\nu_{s-\frac{1}{2}}}\rho_{\rho_1\cdots\rho_{s-\frac{1}{2}}}\ , \tag{4.1.5}$$

where ρ is a symmetric tensor spinor and $\mathcal{G}(\rho)$ is the fermionic analogue of the generalized Einstein tensor (2.2.7), which is invariant under the transformation

$$\delta\rho_{\mu_1\cdots\mu_{s-\frac{1}{2}}} = \partial_{(\mu_1}\eta_{\mu_2\cdots\mu_{s-\frac{1}{2}})}\ . \tag{4.1.6}$$

By substituting the solution (4.1.5) into (4.1.1), we obtain the fermionic NMG-like equations of motion

$$\begin{aligned}\left(\Box - m^2\right)\mathcal{G}_{\mu_1\mu_2\cdots\mu_{s-\frac{1}{2}}}(\rho) &= 0\ ,\\ \gamma^{\mu_1}\mathcal{G}_{\mu_1\mu_2\cdots\mu_{s-\frac{1}{2}}}(\rho) &= 0\ ,\end{aligned} \tag{4.1.7}$$

and by substituting (4.1.5) into (4.1.4), we obtain the fermionic TMG-like equations of motion

$$\begin{aligned}\left(\not\partial - \mu\right)\mathcal{G}_{\mu_1\mu_2\cdots\mu_{s-\frac{1}{2}}}(\rho) &= 0\ ,\\ \gamma^{\mu_1}\mathcal{G}_{\mu_1\mu_2\cdots\mu_{s-\frac{1}{2}}}(\rho) &= 0\ .\end{aligned} \tag{4.1.8}$$

[2] This is different from 4D, in which the Dirac equation is invariant under the parity transformation. See Appendix E for details.

[3] In fact, the $\sqrt{\text{FP}}$-like equations for fermions are more often called the FP equations for fermions.

Both the NMG-like and the TMG-like equations are invariant under (4.1.6).

4.1.3 Actions for the Gravitino

In this subsection, we will show that higher-derivative actions for the gravitino ρ_μ can be constructed without auxiliary fields in a similar way to the bosonic case. Recall that in the kinetic terms of the actions for spin-2 we used the linearized Cotton tensor, which is both divergenceless and traceless. Similarly, for spin-$\frac{3}{2}$ we may construct the "Cottino" (at the linearized level) in the following way

$$\mathcal{C}_\mu(\rho) = \partial\!\!\!/ \mathcal{G}_\mu(\rho) + \varepsilon_\mu{}^{\nu\rho}\partial_\nu \mathcal{G}_\rho(\rho) \ , \tag{4.1.9}$$

so that it can be both divergenceless and γ-traceless:

$$\partial^\mu \mathcal{C}_\mu(\rho) = 0 \quad , \quad \gamma^\mu \mathcal{C}_\mu(\rho) = 0 \ , \tag{4.1.10}$$

and then use it for the kinetic terms of the gravitino.

The spin-$\frac{3}{2}$ NMG-like action can be constructed as

$$S_{\text{NMG spin-}\frac{3}{2}} = \int d^3x \left\{ -\frac{1}{4}\bar{\rho}^\mu \partial\!\!\!/ \mathcal{C}_\mu(\rho) + \frac{m^2}{2}\bar{\rho}^\mu \mathcal{G}_\mu(\rho) \right\} \ . \tag{4.1.11}$$

One may check that the equation of motion of this action reads

$$\frac{1}{2}\partial\!\!\!/ \mathcal{C}_\mu(\rho) - m^2 \mathcal{G}_\mu(\rho) = 0 \ , \tag{4.1.12}$$

whose γ-trace gives the γ-traceless condition:

$$\gamma^\mu \mathcal{G}_\mu(\rho) = 0 \ . \tag{4.1.13}$$

Substituting it back into (4.1.12) leads to the Klein-Gordon equation

$$\left(\Box - m^2\right) \mathcal{G}_\mu(\rho) = 0 \ . \tag{4.1.14}$$

The spin-$\frac{3}{2}$ TMG-like action can be constructed as

$$S_{\text{TMG spin-}\frac{3}{2}} = \int d^3x \left\{ -\frac{1}{4}\bar{\rho}^\mu \mathcal{C}_\mu(\rho) + \frac{\mu}{2}\bar{\rho}^\mu \mathcal{G}_\mu(\rho) \right\} \ . \tag{4.1.15}$$

One may check that the equation of motion of this action reads

$$\frac{1}{2}\mathcal{C}_\mu(\rho) - \mu \mathcal{G}_\mu(\rho) = 0 \ , \tag{4.1.16}$$

whose γ-trace gives the γ-traceless condition:

$$\gamma^\mu \mathcal{G}_\mu(\rho) = 0 \ . \tag{4.1.17}$$

Substituting it back into (4.1.16) leads to the Dirac equation

$$(\partial\!\!\!/ - \mu)\, \mathcal{G}_\mu(\rho) = 0 \ . \tag{4.1.18}$$

One can see that (4.1.12) and (4.1.16) are very similar to their spin-2 counterparts (2.2.15) and (2.4.11). In fact, they can be combined into higher-derivative supersymmetric actions, which has been shown in [22].

4.2 Discussions on Supersymmetric NMG

The actions (2.2.15) and (4.1.11) can be combined into a linearized Supersymmetric New Massive Gravity (SNMG) action

$$\begin{aligned} S_{\text{SNMG}}[h,\rho] \;\; = \;\; & \int d^3x \Big\{ - h^{\mu\nu} G_{\mu\nu}(h) + \frac{1}{m^2} h^{\mu\nu} \varepsilon_\mu{}^{\rho\sigma} \partial_\rho C_{\sigma\nu}(h) \\ & + 4\bar{\rho}^\mu \mathcal{G}_\mu(\rho) - \frac{2}{m^2} \bar{\rho}^\mu \partial\!\!\!/ \mathcal{C}_\mu(\rho) \Big\} \ , \end{aligned} \tag{4.2.1}$$

whose (global) supersymmetry transformation rules are[4]

$$\begin{aligned} \delta h_{\mu\nu} &= \bar{\epsilon}\gamma_{(\mu}\rho_{\nu)} \ , \\ \delta\rho_\mu &= -\frac{1}{4}\gamma^{\rho\sigma}\partial_\rho h_{\mu\sigma}\epsilon \ . \end{aligned} \tag{4.2.2}$$

The non-linear version of this model has also been shown in [22].

In this section, we focus on the discussion on whether we are able to construct another version of the SNMG model which does not contain higher derivatives.

In Subsection 4.2.1, we will first show that by using several auxiliary fields, we are able to construct an action, which is equivalent to (4.2.1), with at most second-order derivatives, and then we will discuss the possibility to do the same construction at the non-linear level. The discussion in Subsection 4.2.1 includes only supersymmetry transformations that close on-shell.

In Subsection 4.2.2 we will first show that the (linearized) lower-derivative action and the corresponding supersymmetry rules can be interpreted as a massless spin-2 multiplet and a massive spin-2 multiplet (both only on-shell), and then we will show how to derive the off-shell version of the massive multiplet. In the end we will show the off-shell version of the linearized action and supersymmetry rules of the lower-derivative SNMG.

[4]In the following, all the SUSY transformation rules only close on-shell. We will not discuss off-shell rules, until Section 4.2.2.

4.2.1 Lowering the Order of Derivatives

4.2.1.1 The Linearized Model

We now try to lower the order of derivatives in the action (4.2.1) and we will also derive the corresponding transformation rules.

We first consider the bosonic part of the action (4.2.1), which can be converted (for the convenience of later discussion at the non-linear level) into

$$S_{\text{bos}}(\text{higher}) = \int d^3x \left\{ -h^{\mu\nu} G_{\mu\nu}(h) + \frac{1}{m^2} R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - \frac{1}{2m^2} R^2 \right\} . \quad (4.2.3)$$

One can lower the order of derivatives by introducing a symmetric auxiliary field $q_{\mu\nu}$, and the resulting equivalent action reads

$$S_{\text{bos}}(\text{lower}) = \int d^3x \left\{ -h^{\mu\nu} G_{\mu\nu}(h) + 2q^{\mu\nu} G_{\mu\nu}(h) - m^2 \left(q^{\mu\nu} q_{\mu\nu} - q^2 \right) \right\} . \quad (4.2.4)$$

The way to show the equivalence is to first derive the equation of motion for q:

$$q_{\mu\nu} = \frac{1}{m^2} G_{\mu\nu}(h) - \frac{1}{2m^2} \eta_{\mu\nu} G^{\text{tr}}(h) , \quad (4.2.5)$$

and then substitute it back into (4.2.4) to eliminate q, which exactly gives (4.2.3).

We next consider the fermionic part of (4.2.1), which can be converted into

$$S_{\text{ferm}}(\text{higher}) = \int d^3x \left\{ 4\bar{\rho}_\mu \gamma^{\mu\nu\rho} \partial_\nu \rho_\rho + \frac{8}{m^2} \bar{\rho}^{\mu\nu} \partial\!\!\!/ \rho_{\mu\nu} - \frac{2}{m^2} \bar{\rho}_{\mu\nu} \gamma^{\mu\nu} \partial\!\!\!/ \gamma^{\rho\sigma} \rho_{\rho\sigma} \right\} , \quad (4.2.6)$$

where we denote $\rho_{\mu\nu} \equiv \partial_{[\mu} \rho_{\nu]}$. To lower the number of derivatives, we first replace the terms that are quadratic in $\rho_{\mu\nu}$ with the kinetic term of an auxiliary field χ_μ, while adding another term with a Lagrange multiplier ψ_μ to fix the relation between $\rho_{\mu\nu}$ and χ_μ:

$$S_{\text{ferm}}(\text{lower}) = \int d^3x \left\{ 4\bar{\rho}_\mu \gamma^{\mu\nu\rho} \partial_\nu \rho_\rho - 4\bar{\chi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \chi_\rho - 8\bar{\psi}_\mu \left(\gamma^{\mu\nu\rho} \rho_{\nu\rho} - m\gamma^{\mu\nu} \chi_\nu \right) \right\} . \quad (4.2.7)$$

The equation of motion for ψ gives

$$\gamma^{\mu\nu\rho} \rho_{\nu\rho} - m\gamma^{\mu\nu} \chi_\nu = 0 , \quad (4.2.8)$$

which enables us to express χ in terms of ρ:

$$\chi_\mu = -\frac{1}{2m} \gamma^{\rho\sigma} \gamma_\mu \rho_{\rho\sigma} , \quad (4.2.9)$$

and consequently gives the following identity

$$-4\bar{\chi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \chi_\rho = \frac{8}{m^2} \bar{\rho}^{\mu\nu} \partial\!\!\!/ \rho_{\mu\nu} - \frac{2}{m^2} \bar{\rho}_{\mu\nu} \gamma^{\mu\nu} \partial\!\!\!/ \gamma^{\rho\sigma} \rho_{\rho\sigma} + (\text{total derivative terms}) . \quad (4.2.10)$$

Using the above formulas to eliminate χ in (4.2.7), one derives (4.2.6), and thus shows their equivalence. Furthermore, one can derive the expression of ψ in terms of χ from the equation of motion for χ

$$\psi_\mu = -\frac{1}{2m}\gamma^{\rho\sigma}\gamma_\mu\chi_{\rho\sigma} \ , \tag{4.2.11}$$

where $\chi_{\rho\sigma} \equiv \partial_{[\rho}\chi_{\sigma]}$.

Adding up (4.2.4) and (4.2.7) we obtain the lower-derivative action for the linearized supersymmetric NMG:

$$\begin{aligned} S_{\rm SNMG}(\text{lower}) \ = \ & \int d^3x \Big\{ -h^{\mu\nu}G_{\mu\nu}(h) + 2q^{\mu\nu}G_{\mu\nu}(h) - m^2\left(q^{\mu\nu}q_{\mu\nu} - q^2\right) \\ & +4\bar\rho_\mu\gamma^{\mu\nu\rho}\partial_\nu\rho_\rho - 4\bar\chi_\mu\gamma^{\mu\nu\rho}\partial_\nu\chi_\rho - 8\bar\psi_\mu\left(\gamma^{\mu\nu\rho}\rho_{\nu\rho} - m\gamma^{\mu\nu}\chi_\nu\right)\Big\} \ , \end{aligned} \tag{4.2.12}$$

The supersymmetry rules for h and ρ are the same as (4.2.2). The rules for the auxiliary fields q, χ and ψ can be derived by substituting (4.2.2) into the variation of (4.2.5), (4.2.9) and (4.2.11), which gives

$$\begin{aligned} \delta q_{\mu\nu} &= \frac{1}{m^2}\bar\epsilon\left(2\delta_{(\mu}{}^\rho\eta_{\nu)\sigma}\gamma_\tau + 2\delta_\rho{}^\sigma\eta_{\tau(\mu}\gamma_{\nu)} - \delta_\rho{}^\sigma\eta_{\mu\nu}\gamma_\tau\right)\partial_\rho\rho^{\sigma\tau} \ , \\ \delta\chi_\mu &= \frac{1}{8m}\gamma^\rho\gamma_\mu\gamma^\sigma\epsilon G_{\rho\sigma}(h) \ , \\ \delta\psi_\mu &= \frac{1}{m^2}\left(\frac{1}{4}\delta_\mu{}^\rho\gamma^{\sigma\nu} - \frac{1}{8}\eta^{\rho\sigma}\gamma_\mu{}^\nu\right)\epsilon\partial_\nu G_{\rho\sigma}(h) \ . \end{aligned} \tag{4.2.13}$$

The above rules are only expressed in terms of h and ρ, one may further convert them by making use of the following equation-of-motion symmetries (schematically):

$$\delta q = (\text{eom of } \psi) \ , \quad \delta\psi = -(\text{eom of } q)$$

and

$$\delta q = (\text{eom of } \chi) \ , \quad \delta\chi = -(\text{eom of } q)$$

in order to replace G with q and to replace ρ with χ and ψ. The resulting supersymmetry rules of (4.2.12) are

$$\begin{aligned} \delta h_{\mu\nu} &= \bar\epsilon\gamma_{(\mu}\rho_{\nu)} \ , \\ \delta\rho_\mu &= -\frac{1}{4}\gamma^{\rho\sigma}\partial_\rho h_{\mu\sigma}\epsilon \ , \\ \delta q_{\mu\nu} &= \bar\epsilon\gamma_{(\mu}\psi_{\nu)} + \frac{1}{m}\bar\epsilon\partial_{(\mu}\chi_{\nu)} \ , \\ \delta\psi_\mu &= -\frac{1}{4}\gamma^{\rho\sigma}\partial_\rho q_{\mu\sigma}\epsilon \ , \\ \delta\chi_\mu &= \frac{m}{4}\gamma^\nu q_{\mu\nu}\epsilon \ . \end{aligned} \tag{4.2.14}$$

4.2.1.2 The Non-Linear Model

We now discuss whether we can lower the order of derivatives for the non-linear SNMG action in [22], in a similar way to the linear case. The non-linear action reads[5]

$$
\begin{aligned}
S_{\mathrm{SNMG}}^{\mathrm{nonlin}}\,(\mathrm{higher}) \;=\; & \int d^3x\; e\Big\{-4R\,(\hat{\omega})+\frac{1}{m^2}R^{\mu\nu ab}\,(\hat{\omega})\,R_{\mu\nu ab}\,(\hat{\omega})-\frac{1}{2m^2}R^2\,(\hat{\omega}) \\
& +4\bar{\rho}_\mu\gamma^{\mu\nu\rho}D_\nu\,(\hat{\omega})\,\rho_\rho+\frac{8}{m^2}\bar{\rho}_{ab}\,(\hat{\omega})\,\not{D}\,(\hat{\omega})\,\rho^{ab}\,(\hat{\omega}) \\
& -\frac{2}{m^2}\bar{\rho}_{\mu\nu}(\hat{\omega})\gamma^{\mu\nu}\not{D}(\hat{\omega})\gamma^{\rho\sigma}\rho_{\rho\sigma}(\hat{\omega}) \\
& -\frac{2}{m^2}R_{\mu\nu ab}\,(\hat{\omega})\,\bar{\rho}_\rho\gamma^{\mu\nu}\gamma^\rho\rho^{ab}\,(\hat{\omega})-\frac{2}{m^2}R\,(\hat{\omega})\,\bar{\rho}^\mu\gamma^\nu\rho_{\mu\nu}\,(\hat{\omega}) \\
& +\text{higher-order fermions and } S\text{-dependent terms}\Big\}\;,
\end{aligned}
\tag{4.2.15}
$$

where $\rho_{\mu\nu}\,(\hat{\omega})\equiv D_{[\mu}\,(\hat{\omega})\,\rho_{\nu]}$. Note that, according to [22], one has to use an extra field S to construct this action. S serves as the auxiliary scalar field in the off-shell multiplet. It does not propagate, but cannot be integrated out at the non-linear level. For simplicity, we consider in the action only the terms that are purely bosonic or bilinear in the fermions, and we ignore terms that contain the field S.[6] Also note that we have replaced the symmetric tensor $h_{\mu\nu}$ with a Dreibein field $e_\mu{}^a$, and the supersymmetric rules (ignoring S) of the action (4.2.15) are

$$
\begin{aligned}
\delta e_\mu{}^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\rho_\mu\;, \\
\delta\rho_\mu &= D_\mu\,(\hat{\omega})\,\epsilon\;.
\end{aligned}
\tag{4.2.16}
$$

We first consider lowering the number of derivatives in the bosonic part of the action. Since the Ricci tensor now depends on a torsion-full spin connection we need a non-symmetric auxiliary tensor $q_{\mu,\nu}$. The action (4.2.15) can then be

[5] Note that in Section 4.2.1.2, the fields $R_{\mu\nu}{}^{ab}$, $\rho_{\mu\nu}$, etc. are non-linear, unless otherwise specified. $\hat{\omega}$ stands for the spin connection with torsion. See Appendix A for details.

[6] Ignoring S does not spoil the consistency of our discussion. In the supersymmetry rules S transforms into a fermion bilinear and in the action the terms that are linear in S only occur with fermion bilinears. Therefore, the supersymmetry transformation of the action contributed by δS contains either S or higher-order fermions, and thus ignoring S together with higher-order fermions is consistent.

converted into the following equivalent action:

$$
\begin{aligned}
S_{\mathrm{SNMG}}^{\mathrm{nonlin}}(\text{higher}) \; = \; & \int d^3x\; e\Big\{ -4R\left(\hat{\omega}\right) - m^2\left(q^{\mu,\nu}q_{\mu,\nu} - q^2\right) + 2q^{\mu,\nu}G_{\mu,\nu}\left(\hat{\omega}\right) \\
& +4\bar{\rho}_\mu\gamma^{\mu\nu\rho}D_\nu\left(\hat{\omega}\right)\rho_\rho + \frac{8}{m^2}\bar{\rho}_{ab}\left(\hat{\omega}\right)\not{D}\left(\hat{\omega}\right)\rho^{ab}\left(\hat{\omega}\right) \\
& -\frac{2}{m^2}\bar{\rho}_{\mu\nu}(\hat{\omega})\gamma^{\mu\nu}\not{D}(\hat{\omega})\gamma^{\rho\sigma}\rho_{\rho\sigma}(\hat{\omega}) \\
& -\frac{2}{m^2}R_{\mu\nu ab}(\hat{\omega})\bar{\rho}_\rho\gamma^{\mu\nu}\gamma^\rho\rho^{ab}(\hat{\omega}) - \frac{2}{m^2}R(\hat{\omega})\bar{\rho}^\mu\gamma^\nu\rho_{\mu\nu}\left(\hat{\omega}\right) \\
& + \text{higher-order fermions and } S\text{-dependent terms}\Big\} \, .
\end{aligned}
\tag{4.2.17}
$$

The equivalence with the previous action can be seen by solving the equation of motion for $q_{\mu,\nu}$

$$
q_{\mu,\nu} = \frac{1}{m^2}G_{\mu,\nu}\left(\hat{\omega}\right) - \frac{1}{2m^2}g_{\mu\nu}G^{\mathrm{tr}}\left(\hat{\omega}\right) \tag{4.2.18}
$$

and substituting this solution back into the action.

We next consider lowering the number of derivatives in the fermionic terms in the action. We first define an auxiliary vector-spinor χ_μ

$$
\chi_\mu = -\frac{1}{2m}\gamma^{\rho\sigma}\gamma_\mu\rho_{\rho\sigma}\left(\hat{\omega}\right) \;, \tag{4.2.19}
$$

which at the linearized level reduces to (4.2.9), and equivalently we have

$$
\rho_{\mu\nu}\left(\hat{\omega}\right) = -m\gamma_{[\mu}\chi_{\nu]} \tag{4.2.20}
$$

or

$$
\gamma^{\mu\nu\rho}\rho_{\nu\rho}\left(\hat{\omega}\right) - m\gamma^{\mu\nu}\chi_\nu = 0 \;. \tag{4.2.21}
$$

Using the definition one can show the following identity

$$
\begin{aligned}
& \frac{8}{m^2}e\bar{\rho}_{ab}(\hat{\omega})\not{D}(\hat{\omega})\rho^{ab}(\hat{\omega}) - \frac{2}{m^2}e\bar{\rho}_{\mu\nu}(\hat{\omega})\gamma^{\mu\nu}\not{D}(\hat{\omega})\left[\gamma^{\rho\sigma}\rho_{\rho\sigma}(\hat{\omega})\right] \\
& \qquad = -4e\bar{\chi}_\mu\gamma^{\mu\nu\rho}D_\nu(\hat{\omega})\chi_\rho - \frac{1}{m}eR_{\mu\nu ab}(\hat{\omega})\bar{\rho}_\rho\gamma^{\mu\nu\rho}\gamma^{ab}\gamma^\sigma\chi_\sigma \\
& \qquad\quad + \text{higher-order fermions and total derivative terms} \;,
\end{aligned}
\tag{4.2.22}
$$

which is the non-linear generalization of the identity (4.2.10). This identity can be used to replace the higher-derivative kinetic terms of the fermions with lower-derivative ones. At the same time we may also use (4.2.20) to replace other $\rho_{\mu\nu}$'s in the action with expressions of χ_μ. Furthermore, we must introduce a Lagrange multiplier ψ_μ whose equation of motion gives (4.2.21), so that after the replacement the new action is equivalent to the old one. This leads to the following action:

$$
\begin{aligned}
I^{\text{nonlin}}_{\text{SNMG}}\,(\text{lower}) \;\; = \;\; & \int d^3x\; e \Big\{ -4R\,(\hat{\omega}) + 2q^{\mu,\nu}G_{\mu,\nu}\,(\hat{\omega}) - m^2\left(q^{\mu,\nu}q_{\mu,\nu} - q^2\right) \\
& +4\bar{\rho}_\mu\gamma^{\mu\nu\rho}D_\nu\,(\hat{\omega})\,\rho_\rho - 4\bar{\chi}_\mu\gamma^{\mu\nu\rho}D_\nu\,(\hat{\omega})\,\chi_\rho \\
& -8\bar{\psi}_\mu\left[\gamma^{\mu\nu\rho}\rho_{\nu\rho}\,(\hat{\omega}) - m\bar{\psi}_\mu\gamma^{\mu\nu}\chi_\nu\right] \\
& -\frac{1}{m}R_{\mu\nu ab}\,(\hat{\omega})\,\bar{\rho}_\rho\gamma^{\mu\nu\rho}\gamma^{ab}\gamma^\sigma\chi_\sigma + \frac{2}{m}R_{\mu\nu ab}\,(\hat{\omega})\,\bar{\rho}_\rho\gamma^{\mu\nu}\gamma^\rho\gamma^a\chi^b \\
& -\frac{1}{m}R\,(\hat{\omega})\,\bar{\rho}_\mu\gamma^{\mu\nu}\chi_\nu - \frac{2}{m}R\,(\hat{\omega})\,\bar{\rho}^\mu\chi_\mu \\
& +\text{higher-order fermions and } S\text{-dependent terms}\Big\}\;. \qquad (4.2.23)
\end{aligned}
$$

Our next task is to derive the supersymmetry rules for the auxiliary fields $q_{\mu,\nu}$, ψ_μ and χ_μ. Using the solutions of the auxiliary fields in terms of $e_\mu{}^a$ and ρ_μ we derived these supersymmetry rules. In this way we obtain a set of supersymmetry rules expressed only in terms of $e_\mu{}^a$ and ρ_μ, which is the non-linear version of (4.2.13). Then by using equation of motion symmetries, like we did in the linearized case, we can obtain a set of rules in analogy to (4.2.14). Due to the fact that the auxiliary fields are not supercovariant, i.e. there are a lot of terms containing the derivative of the parameter field ϵ, the results are rather cumbersome. Since the results we obtained are not illuminating we refrain from giving the explicit expressions here. It would be interesting to see whether a superspace approach could improve on this. Without further insight the lower-derivative formulation of SNMG, if it exists at all at the full non-linear level, does not take the same elegant form as the higher-derivative formulation presented in [22].

4.2.2 Off-Shell Multiplets

In this subsection, our goal is, at the linearized level, to derive the off-shell version of the action (4.2.12) and its supersymmetry rules (4.2.14).

Before discussing the off-shell formalism, we would like to first show that (4.2.12) and the corresponding supersymmetry rules (4.2.14) describe a massless spin-2 multiplet and a massive spin-2 multiplet. The way to show this is to use the following field redefinitions

$$
\begin{aligned}
h_{\mu\nu} &= k_{\mu\nu} + q_{\mu\nu}\;, \\
\rho_\mu &= \lambda_\mu + \psi_\mu\;, \qquad (4.2.24)
\end{aligned}
$$

to replace the fields h and ρ in the action and the transformation rules. The resulting action is

$$\begin{aligned} S_{\text{SNMG}}\left[k,\lambda,q,\chi,\psi\right] &= \int d^3x \Big\{ -k^{\mu\nu}G_{\mu\nu}\left(k\right) + 4\bar{\lambda}_\mu\gamma^{\mu\nu\rho}\partial_\nu\bar{\lambda}_\rho \\ &\quad + q^{\mu\nu}G_{\mu\nu}\left(q\right) - m^2\left(q^{\mu\nu}q_{\mu\nu} - q^2\right) \\ &\quad - 4\bar{\psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - 4\bar{\chi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\chi_\rho + 8m\bar{\psi}_\mu\gamma^{\mu\nu}\chi_\nu \Big\} , \end{aligned} \tag{4.2.25}$$

and the resulting supersymmetry rules are[7]

$$\begin{aligned} \delta k_{\mu\nu} &= \bar{\epsilon}\gamma_{(\mu}\lambda_{\nu)} , \\ \delta\lambda_\mu &= -\frac{1}{4}\gamma^{\rho\sigma}\partial_\rho k_{\mu\sigma}\epsilon , \\ \delta q_{\mu\nu} &= \bar{\epsilon}\gamma_{(\mu}\psi_{\nu)} + \frac{1}{m}\bar{\epsilon}\partial_{(\mu}\chi_{\nu)} , \\ \delta\psi_\mu &= -\frac{1}{4}\gamma^{\rho\sigma}\partial_\rho q_{\mu\sigma}\epsilon , \\ \delta\chi_\mu &= \frac{m}{4}\gamma^\nu q_{\mu\nu}\epsilon . \end{aligned} \tag{4.2.26}$$

Now it is clear that the action and its supersymmetry rules have been divided into two parts: the massless multiplet (k, λ) and the massive multiplet (q, ψ, χ).

The first multiplet consists of a massless graviton k and a massless gravitino λ, which carry gauge symmetries and do not propagate in 3D. One may check that the rules δk and $\delta\lambda$ close only on-shell. The off-shell version of this multiplet has long been known as

$$\begin{aligned} \delta k_{\mu\nu} &= \bar{\epsilon}\gamma_{(\mu}\lambda_{\nu)} , \\ \delta\lambda_\mu &= -\frac{1}{4}\gamma^{\rho\sigma}\partial_\rho k_{\mu\sigma}\epsilon + \frac{1}{2}S\gamma_\mu\epsilon , \\ \delta S &= \frac{1}{4}\bar{\epsilon}\gamma^{\mu\nu}\partial_\mu\lambda_\nu , \end{aligned} \tag{4.2.27}$$

and the corresponding invariant action reads

$$S\left[k,\lambda\right] = \int d^3x \left\{-k^{\mu\nu}G_{\mu\nu}\left(k\right) + 4\bar{\lambda}_\mu\gamma^{\mu\nu\rho}\partial_\nu\bar{\lambda}_\rho + 8S^2\right\} . \tag{4.2.28}$$

The second multiplet consists of a massive graviton q, which on-shell carries two propagating degrees of freedom, and two massive gravitini ψ and χ, which together carry another two propagating degrees of freedom on-shell. The sum and difference

[7]The resulting transformation rule for k is originally $\delta k_{\mu\nu} = \bar{\epsilon}\gamma_{(\mu}\lambda_{\nu)} - \frac{1}{m}\bar{\epsilon}\partial_{(\mu}\chi_{\nu)}$, but the second term on the right-hand-side can be absorbed into the gauge transformation of k, so it has been dropped.

of ψ and χ represent two opposite helicities, which can be shown as follows. One can first derive equations of motion for ψ and χ from the action (4.2.25)

$$\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - m\gamma^{\mu\nu}\chi_\nu = 0 \quad , \quad \gamma^{\mu\nu\rho}\partial_\nu\chi_\rho - m\gamma^{\mu\nu}\psi_\nu = 0 \ , \tag{4.2.29}$$

and by taking the sum and difference of these two equations, one can diagonalize them as

$$\gamma^{\mu\nu\rho}\partial_\nu\zeta^1_\rho - m\gamma^{\mu\nu}\zeta^1_\nu = 0 \quad , \quad \gamma^{\mu\nu\rho}\partial_\nu\zeta^2_\rho + m\gamma^{\mu\nu}\zeta^2_\nu = 0 \ , \tag{4.2.30}$$

where

$$\zeta^1_\mu = \psi_\mu + \chi_\mu \quad , \quad \zeta^2_\mu = \psi_\mu - \chi_\mu \ . \tag{4.2.31}$$

Then one can check that (4.2.30) is equivalent to:

$$\begin{aligned} &\left(\partial\!\!\!/ + m\right)\zeta^1_\mu = 0 \ , \ \gamma^\mu\zeta^1_\mu = 0 \ , \ \partial^\mu\zeta^1_\mu = 0 \ , \\ &\left(\partial\!\!\!/ - m\right)\zeta^2_\mu = 0 \ , \ \gamma^\mu\zeta^2_\mu = 0 \ , \ \partial^\mu\zeta^2_\mu = 0 \ , \end{aligned} \tag{4.2.32}$$

which is the spin-$\frac{3}{2}$ case of (4.1.4) describing two helicities.

Now there are two questions about this massive multiplet. First, one may check that the rules δq, $\delta\psi$ and $\delta\chi$ in (4.2.26) close only on-shell, so what does the off-shell multiplet look like? Second, due to the fact that m appears as a denominator in δq, if we take the massless limit ($m \to 0$), we do not directly see that the massive multiplet may reduce to a massless multiplet, so how should we exactly take the massless limit to bridge the two multiplets? Both questions will be answered in the following part of this section. To make it pedagogical, we will first discuss the massive spin-1 (Proca) off-shell multiplet as a toy model, then we will introduce the spin-2 (Fierz-Pauli) off-shell multiplet in a similar manner.

4.2.2.1 Supersymmetric Proca

The way we obtain a 3D off-shell massive multiplet is to do the Kaluza–Klein (KK) reduction of the corresponding known 4D off-shell massless multiplet. To illustrate this, in the following we show how to obtain the 3D supersymmetric Proca theory from the KK reduction of an off-shell 4D $\mathcal{N} = 1$ supersymmetric Maxwell theory and a subsequent truncation to the first massive KK sector. This is a warming-up exercise for the spin-2 case which will be discussed after the spin-1.

Kaluza–Klein reduction

Our starting point is the 4D $\mathcal{N} = 1$ supersymmetric Maxwell multiplet which consists of a vector $\hat{V}_{\hat{\mu}}$, a 4-component Majorana spinor $\hat{\psi}$ and a real auxiliary scalar $\hat{F}$. In this section we indicate 4D indices and fields depending on the 4D coordinates with hats. We do not indicate spinor indices. The supersymmetry rules, with a constant 4-component Majorana spinor parameter ϵ, and gauge transformation, with local parameter $\hat{\Lambda}$, of these fields are given by

$$\begin{aligned} \delta\hat{V}_{\hat{\mu}} &= -\bar{\epsilon}\Gamma_{\hat{\mu}}\hat{\psi} + \partial_{\hat{\mu}}\hat{\Lambda} \, , \\ \delta\hat{\psi} &= \frac{1}{8}\Gamma^{\hat{\mu}\hat{\nu}}\hat{F}_{\hat{\mu}\hat{\nu}}\epsilon + \frac{1}{4}i\Gamma_5\hat{F}\epsilon \, , \\ \delta\hat{F} &= i\bar{\epsilon}\Gamma_5\Gamma^{\hat{\mu}}\partial_{\hat{\mu}}\hat{\psi} \, , \end{aligned} \tag{4.2.33}$$

where $\hat{F}_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}}\hat{V}_{\hat{\nu}} - \partial_{\hat{\nu}}\hat{V}_{\hat{\mu}}$, and we denote the gamma matrices in 4D by the capital letter Γ.

In the following, we will split the 4D coordinates as $x^{\hat{\mu}} = (x^\mu, x^3)$, where x^3 denotes the compactified circle coordinate. Since all fields are periodic in x^3, we can write them as a Fourier series. For example:

$$\hat{V}_{\hat{\mu}}(x^{\hat{\mu}}) = \sum_n V_{\hat{\mu},n}(x^\mu)e^{inmx^3} \, , \qquad n \in \mathbb{Z} \, , \tag{4.2.34}$$

where $m \neq 0$ has mass dimensions and corresponds to the inverse circle radius. The Fourier coefficients $V_{\hat{\mu},n}(x^\mu)$ correspond to 3D (un-hatted) fields. We first consider the bosonic fields. The reality condition on the 4D vector and scalar implies that only the 3D ($n = 0$) zero modes are real. All other modes are complex but only the positive ($n \geq 1$) modes are independent, since

$$V_{\hat{\mu},-n} = V^\star_{\hat{\mu},n} \, , \qquad F_{-n} = F^\star_n \, , \qquad n \neq 0 \, . \tag{4.2.35}$$

In the following we will be mainly interested in the $n = 1$ modes whose real and imaginary parts we indicate by

$$\begin{aligned} V^{(1)}_\mu &\equiv \frac{1}{2}\big(V_{\mu,1} + V^\star_{\mu,1}\big) \, , & V^{(2)}_\mu &\equiv \frac{1}{2i}\big(V_{\mu,1} - V^\star_{\mu,1}\big) \, , \\ \phi^{(1)} &\equiv \frac{1}{2}\big(V_{3,1} + V^\star_{3,1}\big) \, , & \phi^{(2)} &\equiv \frac{1}{2i}\big(V_{3,1} - V^\star_{3,1}\big) \, , \\ F^{(1)} &\equiv \frac{1}{2}\big(F_1 + F^\star_1\big) \, , & F^{(2)} &\equiv \frac{1}{2i}\big(F_1 - F^\star_1\big) \, . \end{aligned} \tag{4.2.36}$$

Similarly, the Majorana condition of the 4D spinor $\hat{\psi}$ implies that the $n = 0$ mode is Majorana but that the independent positive ($n \geq 1$) modes are Dirac. This is equivalent to two (4-component, 3D reducible) Majorana spinors which we indicate by

$$\psi^{(1)} = \frac{1}{2}\big(\psi_1 + B^{-1}\psi^\star_1\big) \, , \qquad \psi^{(2)} = \frac{1}{2i}\big(\psi_1 - B^{-1}\psi^\star_1\big) \, . \tag{4.2.37}$$

Here B is the 4×4 matrix $B = iC\Gamma_0$, where C is the 4×4 charge conjugation matrix, and one may check that both $\psi^{(1)}$ and $\psi^{(2)}$ satisfy the Majorana condition $\psi^{(1,2)\star} = B\psi^{(1,2)}$.

Substituting the harmonic expansion (4.2.34) of the fields and a similar expansion of the gauge parameter $\hat{\Lambda}$ into the transformation rules (4.2.33), we find the following transformation rules for the first ($n = 1$) KK modes:

$$\begin{aligned}
\delta\phi^{(1)} &= -\bar{\epsilon}\Gamma_3\psi^{(1)} - m\Lambda^{(2)} \ , \\
\delta\phi^{(2)} &= -\bar{\epsilon}\Gamma_3\psi^{(2)} + m\Lambda^{(1)} \ , \\
\delta V_\mu^{(1)} &= -\bar{\epsilon}\Gamma_\mu\psi^{(1)} + \partial_\mu\Lambda^{(1)} \ , \\
\delta V_\mu^{(2)} &= -\bar{\epsilon}\Gamma_\mu\psi^{(2)} + \partial_\mu\Lambda^{(2)} \ , \\
\delta F^{(1)} &= i\bar{\epsilon}\Gamma_5\Gamma^\mu\partial_\mu\psi^{(1)} - im\bar{\epsilon}\Gamma_5\Gamma_3\psi^{(2)} \ , \\
\delta F^{(2)} &= i\bar{\epsilon}\Gamma_5\Gamma^\mu\partial_\mu\psi^{(2)} + im\bar{\epsilon}\Gamma_5\Gamma_3\psi^{(1)} \ , \\
\delta\psi^{(1)} &= \frac{1}{8}\Gamma^{\mu\nu}F_{\mu\nu}^{(1)}\epsilon + \frac{1}{4}\Gamma^\mu\Gamma_3\partial_\mu\phi^{(1)}\epsilon + \frac{i}{4}\Gamma_5F^{(1)}\epsilon + \frac{m}{4}\Gamma^\mu\Gamma_3V_\mu^{(2)}\epsilon \ , \\
\delta\psi^{(2)} &= \frac{1}{8}\Gamma^{\mu\nu}F_{\mu\nu}^{(2)}\epsilon + \frac{1}{4}\Gamma^\mu\Gamma_3\partial_\mu\phi^{(2)}\epsilon + \frac{i}{4}\Gamma_5F^{(2)}\epsilon - \frac{m}{4}\Gamma^\mu\Gamma_3V_\mu^{(1)}\epsilon \ ,
\end{aligned} \tag{4.2.38}$$

where we have defined

$$\Lambda^{(1)} = \frac{1}{2}\big(\Lambda_1 + \Lambda_1^\star\big) \ , \qquad \Lambda^{(2)} = \frac{1}{2i}\big(\Lambda_1 - \Lambda_1^\star\big) \ . \tag{4.2.39}$$

Note that apart from global supersymmetry transformations with parameter ϵ and gauge transformations with parameters $\Lambda^{(1)}$, $\Lambda^{(2)}$, there is another symmetry, which is a global SO(2) transformation that rotates the real and imaginary parts of the 3D fields:

$$\begin{aligned}
\delta\phi^{(1)} &= -m\xi\phi^{(2)} \ , \quad \delta\phi^{(2)} = m\xi\phi^{(1)} \ , \\
\delta V_\mu^{(1)} &= -m\xi V_\mu^{(2)} \ , \quad \delta V_\mu^{(2)} = m\xi V_\mu^{(1)} \ , \\
\delta F^{(1)} &= -m\xi F^{(2)} \ , \quad \delta F^{(2)} = m\xi F^{(1)} \ , \\
\delta\psi^{(1)} &= -m\xi\psi^{(2)} \ , \quad \delta\psi^{(2)} = m\xi\psi^{(1)} \ ,
\end{aligned} \tag{4.2.40}$$

where ξ is the parameter. This SO(2) transformation corresponds to a central charge transformation and is a remnant of the translation in the compact circle direction. One may check the closure of (4.2.38) by calculating the commutator of two supersymmetry transformations acting on a field, which results in not only the translation in the three non-compact dimensions, but also (4.2.40).

In order to write the 3D 4-component Majorana spinors in terms of two irreducible 2-component Majorana spinors it is convenient to choose the following representation of the Γ-matrices in terms of 2 × 2 block matrices:

$$\Gamma_\mu = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & -\gamma_\mu \end{pmatrix} \ , \quad \Gamma_3 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \ , \quad \Gamma_5 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ . \tag{4.2.41}$$

The 3D 2×2 matrices γ_μ satisfy the standard relations $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$ and can be chosen explicitly in terms of the Pauli matrices by

$$\gamma_\mu = (i\sigma_1, \sigma_2, \sigma_3)\,. \tag{4.2.42}$$

In this representation the 4D charge conjugation matrix C is given by

$$C = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}\,, \tag{4.2.43}$$

where

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.2.44}$$

is the 3D charge conjugation matrix.

Using the above representation the 4-component Majorana spinors decompose into two 3D irreducible Majorana spinors as follows:

$$\psi^{(1)} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}\,, \qquad \psi^{(2)} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\,, \qquad \epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}\,. \tag{4.2.45}$$

In terms of these 2-component spinors the transformation rules (4.2.38) and (4.2.40) read

$$\begin{aligned}
\delta\phi^{(1)} &= -\bar\epsilon_1\chi_2 + \bar\epsilon_2\chi_1 - m\Lambda^{(2)} - m\xi\phi^{(2)}\ , \\
\delta\phi^{(2)} &= -\bar\epsilon_1\psi_2 + \bar\epsilon_2\psi_1 + m\Lambda^{(1)} + m\xi\phi^{(1)}\ , \\
\delta V_\mu^{(1)} &= -\bar\epsilon_1\gamma_\mu\chi_1 - \bar\epsilon_2\gamma_\mu\chi_2 + \partial_\mu\Lambda^{(1)} - m\xi V_\mu^{(2)}\ , \\
\delta V_\mu^{(2)} &= -\bar\epsilon_1\gamma_\mu\psi_1 - \bar\epsilon_2\gamma_\mu\psi_2 + \partial_\mu\Lambda^{(2)} + m\xi V_\mu^{(1)}\ , \\
\delta F^{(1)} &= -\bar\epsilon_1\gamma^\mu\partial_\mu\chi_2 + \bar\epsilon_2\gamma^\mu\partial_\mu\chi_1 - m(\bar\epsilon_1\psi_1 + \bar\epsilon_2\psi_2) - m\xi F^{(2)}\ , \\
\delta F^{(2)} &= -\bar\epsilon_1\gamma^\mu\partial_\mu\psi_2 + \bar\epsilon_2\gamma^\mu\partial_\mu\psi_1 + m(\bar\epsilon_1\chi_1 + \bar\epsilon_2\chi_2) + m\xi F^{(1)}\ , \\
\delta\chi_1 &= \frac{1}{8}\gamma^{\mu\nu}F_{\mu\nu}^{(1)}\epsilon_1 + \frac{1}{4}\big(\gamma^\mu\partial_\mu\phi^{(1)} + F^{(1)} + m\gamma^\mu V_\mu^{(2)}\big)\epsilon_2 - m\xi\psi_1\ , \\
\delta\chi_2 &= \frac{1}{8}\gamma^{\mu\nu}F_{\mu\nu}^{(1)}\epsilon_2 - \frac{1}{4}\big(\gamma^\mu\partial_\mu\phi^{(1)} + F^{(1)} + m\gamma^\mu V_\mu^{(2)}\big)\epsilon_1 - m\xi\psi_2\ , \\
\delta\psi_1 &= \frac{1}{8}\gamma^{\mu\nu}F_{\mu\nu}^{(2)}\epsilon_1 + \frac{1}{4}\big(\gamma^\mu\partial_\mu\phi^{(2)} + F^{(2)} - m\gamma^\mu V_\mu^{(1)}\big)\epsilon_2 + m\xi\chi_1\ , \\
\delta\psi_2 &= \frac{1}{8}\gamma^{\mu\nu}F_{\mu\nu}^{(2)}\epsilon_2 - \frac{1}{4}\big(\gamma^\mu\partial_\mu\phi^{(2)} + F^{(2)} - m\gamma^\mu V_\mu^{(1)}\big)\epsilon_1 + m\xi\chi_2\ ,
\end{aligned} \tag{4.2.46}$$

where $F_{\mu\nu}^{(1,2)} = \partial_\mu V_\nu^{(1,2)} - \partial_\nu V_\mu^{(1,2)}$.

Truncation

In the process of KK reduction, the number of supercharges stays the same. The 3D multiplet (4.2.46) we found previously thus exhibits four supercharges and hence corresponds to an $\mathcal{N}=2$ multiplet with a central charge transformation, containing two vectors. One can however truncate it to an $\mathcal{N}=1$ multiplet, not subjected to a central charge transformation and containing only one vector. This truncated multiplet will be the starting point to obtain an $\mathcal{N}=1$ supersymmetric version of the Proca theory. The $\mathcal{N}=1$ truncation is given by:

$$\phi^{(2)} = V_\mu^{(1)} = F^{(2)} = \chi_2 = \psi_1 = 0 \;, \tag{4.2.47}$$

provided that at the same time we truncate the following symmetries:

$$\epsilon_1 = \Lambda^{(1)} = \xi = 0 \;. \tag{4.2.48}$$

Substituting this truncation into the transformation rules (4.2.46), we find the following $\mathcal{N}=1$ massive vector supermultiplet:

$$\begin{aligned}
\delta\phi^{(1)} &= \bar\epsilon_2\chi_1 - m\Lambda^{(2)} \;,\\
\delta V_\mu^{(2)} &= -\bar\epsilon_2\gamma_\mu\psi_2 + \partial_\mu\Lambda^{(2)} \;,\\
\delta\psi_2 &= \frac{1}{8}\gamma^{\mu\nu}F_{\mu\nu}^{(2)}\epsilon_2 \;,\\
\delta\chi_1 &= \frac{1}{4}\left(\gamma^\mu\partial_\mu\phi^{(1)} + F^{(1)} + m\gamma^\mu V_\mu^{(2)}\right)\epsilon_2 \;,\\
\delta F^{(1)} &= \bar\epsilon_2\gamma^\mu\partial_\mu\chi_1 - m\bar\epsilon_2\psi_2 \;.
\end{aligned} \tag{4.2.49}$$

Redefining $\epsilon_2 \to \epsilon\,, \Lambda^{(2)} \to \Lambda$ and

$$V_\mu^{(2)} \to V_\mu \;, \quad \phi^{(1)} \to 4\phi\,, \quad F^{(1)} \to -F \;, \quad \psi_2 \to \psi\,, \quad \chi_1 \to \chi \quad \text{and} \quad m \to 4m \;, \tag{4.2.50}$$

we obtain

$$\begin{aligned}
\delta\phi &= \frac{1}{4}\bar\epsilon\chi - m\Lambda \;,\\
\delta V_\mu &= -\bar\epsilon\gamma_\mu\psi + \partial_\mu\Lambda \;,\\
\delta\psi &= \frac{1}{8}\gamma^{\mu\nu}F_{\mu\nu}\epsilon \;,\\
\delta\chi &= \gamma^\mu\left(\partial_\mu\phi + mV_\mu\right)\epsilon - \frac{1}{4}F\epsilon \;,\\
\delta F &= -\bar\epsilon\gamma^\mu\partial_\mu\chi + 4m\bar\epsilon\psi \;,
\end{aligned} \tag{4.2.51}$$

where $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$. The transformation rules (4.2.51) leave the following action invariant:

$$\begin{aligned}
S_{\text{SUSY Proca'}} = \int d^3x\Big\{ &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\left(\partial_\mu\phi + mV_\mu\right)\left(\partial_\mu\phi + mV_\mu\right)\\
&-2\bar\psi\partial\!\!\!/\psi - \frac{1}{8}\bar\chi\partial\!\!\!/\chi + m\,\bar\psi\chi + \frac{1}{32}F^2\Big\} \;.
\end{aligned} \tag{4.2.52}$$

The gauge transformation with parameter Λ is a Stückelberg symmetry, that can be fixed by imposing the gauge condition

$$\phi = \text{const} \ . \tag{4.2.53}$$

Taking the resulting compensating gauge transformation

$$\Lambda = \frac{1}{4m}\,\bar{\epsilon}\chi \tag{4.2.54}$$

into account, we obtain the final form of the supersymmetry transformation rules of the $\mathcal{N}=1$ supersymmetric Proca theory:

$$\begin{aligned}
\delta V_\mu &= -\bar{\epsilon}\gamma_\mu\psi + \frac{1}{4m}\bar{\epsilon}\partial_\mu\chi \ , \\
\delta\psi &= \frac{1}{8}\gamma^{\mu\nu}F_{\mu\nu}\epsilon \ , \\
\delta\chi &= m\gamma^\mu\epsilon V_\mu - \frac{1}{4}F\epsilon \ , \\
\delta F &= -\bar{\epsilon}\gamma^\mu\partial_\mu\chi + 4m\bar{\epsilon}\psi \ .
\end{aligned} \tag{4.2.55}$$

The supersymmetric Proca action is then given by

$$\begin{aligned}
&S_{\text{SUSY Proca}} \\
&= \int d^3x \left\{ -\frac{1}{4}F_{\mu\nu}\,F^{\mu\nu} - \frac{1}{2}\,m^2 V_\mu\,V^\mu - 2\bar{\psi}\partial\!\!\!/\psi - \frac{1}{8}\bar{\chi}\partial\!\!\!/\chi + m\,\bar{\psi}\chi + \frac{1}{32}F^2 \right\} \ .
\end{aligned} \tag{4.2.56}$$

This finishes our description of how to obtain the 3D off-shell massive $\mathcal{N}=1$ vector multiplet from a KK reduction and subsequent truncation onto the first massive KK sector of the 4D off-shell massless $\mathcal{N}=1$ vector multiplet.

Massless limit

Due to the factors of $1/m$ in the transformation rules (4.2.55) it is non-trivial to take the massless limit $m \to 0$. To take this limit one should first go back to the formulation (4.2.51) with the Stückelberg symmetry, whose massless limit is well-defined. This is equivalent to making the following redefinition in (4.2.55):

$$V_\mu = \tilde{V}_\mu + \frac{1}{m}\partial_\mu\phi \ , \tag{4.2.57}$$

and then renaming $\tilde{V}$ to V.

Taking the $m \to 0$ limit of the supersymmetry rules in (4.2.51), we thus obtain the rules of a massless vector multiplet (V_μ, ψ):

$$\begin{aligned}
\delta V_\mu &= -\bar{\epsilon}\gamma_\mu\psi \ , \\
\delta\psi &= \frac{1}{8}\gamma^{\mu\nu}\epsilon\,F_{\mu\nu} \ ,
\end{aligned} \tag{4.2.58}$$

and the rules of a massless scalar multiplet (ϕ, χ, F):

$$\begin{aligned}
\delta\phi &= \frac{1}{4}\bar{\epsilon}\chi \ , \\
\delta\chi &= \gamma^{\mu}\epsilon\,(\partial_{\mu}\phi) - \frac{1}{4}F\epsilon \ , \\
\delta F &= -\bar{\epsilon}\gamma^{\mu}\partial_{\mu}\chi \ .
\end{aligned} \tag{4.2.59}$$

The redefinition (4.2.57) can also bring the action (4.2.56) back to the action (4.2.52) with the Stückelberg symmetry. The $m \to 0$ limit of the action (4.2.52) gives

$$S_{\text{SUSY Proca'}\ m\to 0} = \int d^3x \left\{ \left(-\frac{1}{4}F_{\mu\nu}\,F^{\mu\nu} - 2\bar{\psi}\partial\!\!\!/\psi \right) + \left(-\frac{1}{2}\partial_{\mu}\phi\,\partial^{\mu}\phi - \frac{1}{8}\bar{\chi}\partial\!\!\!/\chi + \frac{1}{32}F^2 \right) \right\} \ , \tag{4.2.60}$$

which is the sum of the supersymmetric massless vector and scalar actions, which are invariant under (4.2.58) and (4.2.59), respectively.

4.2.2.2 Supersymmetric Fierz–Pauli

Now we extend the discussion to the spin-2 case, skipping some of the details we explained in the spin-1 case. We use the same notation.

Kaluza–Klein reduction and truncation

Our starting point is the off-shell 4D $\mathcal{N} = 1$ massless spin-2 multiplet which consists of a symmetric tensor $\hat{q}_{\hat{\mu}\hat{\nu}}$, a gravitino $\hat{\psi}_{\hat{\mu}}$, an auxiliary vector $\hat{A}_{\hat{\mu}}$ and two auxiliary scalars $\hat{M}$ and $\hat{N}$. This corresponds to the linearized version of the "old minimal supergravity" multiplet. The supersymmetry rules, with constant spinor parameter ϵ, and gauge transformations of these fields, with local vector parameter $\hat{\Lambda}_{\hat{\mu}}$ and local spinor parameter $\hat{\eta}$, are given by [45, 46]: [8]

$$\begin{aligned}
\delta\hat{q}_{\hat{\mu}\hat{\nu}} &= \bar{\epsilon}\,\Gamma_{(\hat{\mu}}\,\hat{\psi}_{\hat{\nu})} + \partial_{(\hat{\mu}}\,\hat{\Lambda}_{\hat{\nu})} \ , \\
\delta\hat{\psi}_{\hat{\mu}} &= -\frac{1}{4}\Gamma^{\hat{\rho}\hat{\lambda}}\,\partial_{\hat{\rho}}\hat{q}_{\hat{\lambda}\hat{\mu}}\epsilon \\
&\quad -\frac{1}{12}\Gamma_{\hat{\mu}}(\hat{M} + i\,\Gamma_5\hat{N})\,\epsilon + \frac{1}{4}i\,\hat{A}_{\hat{\mu}}\Gamma_5\epsilon - \frac{1}{12}i\,\Gamma_{\hat{\mu}}\Gamma^{\hat{\rho}}\hat{A}_{\hat{\rho}}\Gamma_5\epsilon + \partial_{\hat{\mu}}\hat{\eta} \ , \\
\delta\hat{M} &= -\bar{\epsilon}\,\Gamma^{\hat{\rho}\hat{\lambda}}\,\partial_{\hat{\rho}}\hat{\psi}_{\hat{\lambda}} \ , \\
\delta\hat{N} &= -i\,\bar{\epsilon}\,\Gamma_5\,\Gamma^{\hat{\rho}\hat{\lambda}}\,\partial_{\hat{\rho}}\hat{\psi}_{\hat{\lambda}} \ , \\
\delta\hat{A}_{\hat{\mu}} &= \frac{3}{2}i\,\bar{\epsilon}\,\Gamma_5\,\Gamma_{\hat{\mu}}{}^{\hat{\rho}\hat{\lambda}}\,\partial_{\hat{\rho}}\,\hat{\psi}_{\hat{\lambda}} - i\,\bar{\epsilon}\,\Gamma_5\,\Gamma_{\hat{\mu}}\Gamma^{\hat{\rho}\hat{\lambda}}\,\partial_{\hat{\rho}}\hat{\psi}_{\hat{\lambda}} \ .
\end{aligned} \tag{4.2.61}$$

[8] $\Gamma_5 = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3$.

Like in the spin-1 case we first perform a harmonic expansion of all fields and local parameters and substitute these into the transformation rules (4.2.61). Projecting onto the lowest KK massive sector we then obtain all the transformation rules of the real and imaginary parts of the $n = 1$ modes, like in (4.2.38) for the spin-1 case. We indicate the real and imaginary parts of the bosonic modes by:

$$\begin{aligned}
q^{(1)}_{\mu\nu} &\equiv \frac{1}{2}\left(q_{\mu\nu,1} + q^\star_{\mu\nu,1}\right) \quad , \quad q^{(2)}_{\mu\nu} \equiv \frac{1}{2i}\left(q_{\mu\nu,1} - q^\star_{\mu\nu,1}\right) \ , \\
V^{(1)}_{\mu} &\equiv \frac{1}{2}\left(q_{\mu 3,1} + q^\star_{\mu 3,1}\right) \quad , \quad V^{(2)}_{\mu} \equiv \frac{1}{2i}\left(q_{\mu 3,1} - q^\star_{\mu 3,1}\right) \ , \\
\phi^{(1)} &\equiv \frac{1}{2}\left(q_{33,1} + q^\star_{33,1}\right) \quad , \quad \phi^{(2)} \equiv \frac{1}{2i}\left(q_{33,1} - q^\star_{33,1}\right) \ , \\
M^{(1)} &\equiv \frac{1}{2}\left(M + M^\star\right) \quad , \quad M^{(2)} \equiv \frac{1}{2i}\left(M - M^\star\right) \ , \\
N^{(1)} &\equiv \frac{1}{2}\left(N + N^\star\right) \quad , \quad N^{(2)} \equiv \frac{1}{2i}\left(N - N^\star\right) \ , \\
A^{(1)}_{\mu} &\equiv \frac{1}{2}\left(A_{\mu,1} + A^\star_{\mu,1}\right) \quad , \quad A^{(2)}_{\mu} \equiv \frac{1}{2i}\left(A_{\mu,1} - A^\star_{\mu,1}\right) \ , \\
P^{(1)} &\equiv \frac{1}{2}\left(A_{3,1} + A^\star_{3,1}\right) \quad , \quad P^{(2)} \equiv \frac{1}{2i}\left(A_{3,1} - A^\star_{3,1}\right) \ ,
\end{aligned} \tag{4.2.62}$$

while the fermionic modes decompose into two Majorana modes:

$$\begin{aligned}
\psi^{(1)}_{\mu} &\equiv \frac{1}{2}\left(\psi_{\mu,1} + B^{-1}\psi^\star_{\mu,1}\right) \ , \qquad \psi^{(2)}_{\mu} \equiv \frac{1}{2i}\left(\psi_{\mu,1} - B^{-1}\psi^\star_{\mu,1}\right) \ , \\
\psi^{(1)}_{3} &\equiv \frac{1}{2}\left(\psi_{3,1} + B^{-1}\psi^\star_{3,1}\right) \ , \qquad \psi^{(2)}_{3} \equiv \frac{1}{2i}\left(\psi_{3,1} - B^{-1}\psi^\star_{3,1}\right) \ .
\end{aligned} \tag{4.2.63}$$

We next use the representation (4.2.41) of the Γ-matrices and decompose the 4-component spinors into two 2-component spinors as follows:[9]

$$\begin{aligned}
&\psi^{(1)}_{\mu} = \begin{pmatrix} \psi_{\mu 1} \\ \psi_{\mu 2} \end{pmatrix} , \qquad \psi^{(1)}_{3} = \begin{pmatrix} \chi_{1} \\ \chi_{2} \end{pmatrix} , \\
&\psi^{(2)}_{\mu} = \begin{pmatrix} \chi_{\mu 1} \\ \chi_{\mu 2} \end{pmatrix} , \qquad \psi^{(2)}_{3} = \begin{pmatrix} \psi_{1} \\ \psi_{2} \end{pmatrix} , \\
&\eta^{(1)} = \begin{pmatrix} \eta^{(1)}_{1} \\ \eta^{(1)}_{2} \end{pmatrix} , \qquad \eta^{(2)} = \begin{pmatrix} \eta^{(2)}_{1} \\ \eta^{(2)}_{2} \end{pmatrix} , \qquad \epsilon = \begin{pmatrix} \epsilon_{1} \\ \epsilon_{2} \end{pmatrix} .
\end{aligned} \tag{4.2.64}$$

Furthermore, we perform the following consistent truncation of the fields

$$\phi^{(2)} = V^{(1)}_{\mu} = q^{(2)}_{\mu\nu} = M^{(2)} = N^{(1)} = P^{(2)} = A^{(1)}_{\mu} = \chi_2 = \psi_1 = \psi_{\mu 1} = \chi_{\mu 2} = 0$$

and of the parameters

$$\Lambda^{(2)}_{\mu} = \Lambda^{(1)}_{3} = \epsilon_1 = \eta^{(1)}_{1} = \eta^{(2)}_{2} = \xi = 0 \ .$$

[9]Note that in Section 4.2.2.2 the notation ψ_μ and ψ stand for two independent fields of different types. So do χ_μ and χ. The same applies when they carry numerical labels.

For simplicity, from now on we drop all numerical upper indices, e.g. $\phi^{(1)} = \phi$, and all numerical lower indices, e.g. $\psi_{\mu 1} = \psi_\mu$ of the remaining non-zero fields (but not of the parameters). We find that the transformation rules for these fields under supersymmetry, with constant 2-component spinor parameter ϵ, and Stückelberg symmetries, with local scalar and vector parameters $\Lambda_3\,, \Lambda_\mu\,,$ and 2-component spinor parameters η_1 and η_2, are given by [10]

$$
\begin{aligned}
\delta q_{\mu\nu} &= \bar\epsilon\gamma_{(\mu}\psi_{\nu)} + \partial_{(\mu}\Lambda_{\nu)}\ ,\\
\delta V_\mu &= \frac{1}{2}\bar\epsilon\gamma_\mu\psi - \frac{1}{2}\bar\epsilon\chi_\mu + \frac{1}{2}\partial_\mu\Lambda_3 + \frac{1}{2}m\Lambda_\mu\ ,\\
\delta\phi &= -\bar\epsilon\,\chi - m\Lambda_3\ ,\\
\delta\psi_\mu &= -\frac{1}{4}\gamma^{\rho\lambda}\partial_\rho q_{\lambda\mu}\epsilon + \frac{1}{12}\gamma_\mu M\epsilon + \frac{1}{12}\gamma_\mu P\epsilon + \partial_\mu\eta_2\ ,\\
\delta\psi &= -\frac{1}{4}\gamma^{\rho\lambda}\partial_\rho V_\lambda\epsilon - \frac{1}{12}N\epsilon - \frac{1}{12}\gamma^\rho A_\rho\epsilon + m\eta_2\ ,\\
\delta\chi_\mu &= -\frac{1}{4}\gamma^\rho\partial_\rho V_\mu\epsilon + \frac{1}{4}m\gamma^\rho q_{\rho\mu}\epsilon - \frac{1}{12}\gamma_\mu N\epsilon + \frac{1}{4}A_\mu\epsilon - \frac{1}{12}\gamma_\mu\gamma^\rho A_\rho\epsilon + \partial_\mu\eta_1\ ,\\
\delta\chi &= -\frac{1}{4}\gamma^\rho\partial_\rho\phi\epsilon - \frac{1}{12}M\epsilon + \frac{1}{6}P\epsilon - \frac{1}{4}m\gamma^\rho V_\rho\epsilon - m\eta_1\ ,\\
\delta M &= -\bar\epsilon\gamma^\rho\partial_\rho\chi + \bar\epsilon\gamma^{\rho\lambda}\partial_\rho\psi_\lambda - m\bar\epsilon\gamma^\rho\chi_\rho\ ,\\
\delta N &= -\bar\epsilon\gamma^\rho\partial_\rho\psi - \bar\epsilon\gamma^{\rho\lambda}\partial_\rho\chi_\lambda + m\bar\epsilon\gamma^\rho\psi_\rho\ ,\\
\delta P &= \bar\epsilon\gamma^\rho\partial_\rho\chi + \frac{1}{2}\bar\epsilon\gamma^{\rho\lambda}\partial_\rho\psi_\lambda + m\bar\epsilon\gamma^\rho\chi_\rho\ ,\\
\delta A_\mu &= \frac{3}{2}\bar\epsilon\gamma_\mu{}^{\rho\lambda}\partial_\rho\chi_\lambda - \bar\epsilon\gamma_\mu\gamma^{\rho\lambda}\partial_\rho\chi_\lambda + \frac{1}{2}\bar\epsilon\gamma_\mu{}^\rho\partial_\rho\psi - \bar\epsilon\partial_\mu\psi - \frac{1}{2}m\bar\epsilon\gamma_\mu{}^\rho\psi_\rho + m\bar\epsilon\psi_\mu\ .
\end{aligned}
\tag{4.2.65}
$$

The action invariant under the transformations (4.2.65) is given by

$$
\begin{aligned}
S_{\text{SUSY FP'}} = \int d^3x\Big\{ & q^{\mu\nu}G_{\mu\nu}(q) - m^2\left(q^{\mu\nu}q_{\mu\nu} - q^2\right)\\
& + 2q^{\mu\nu}\partial_\mu\partial_\nu\phi - 2q\partial^\alpha\partial_\alpha\phi - F^{\mu\nu}F_{\mu\nu} + 4mq^{\mu\nu}\partial_{(\mu}V_{\nu)} - 4mq\partial^\mu V_\mu\\
& - 4\bar\psi_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - 4\bar\chi_\mu\gamma^{\mu\nu\rho}\partial_\nu\chi_\rho + 8m\bar\psi_\mu\gamma^{\mu\nu}\chi_\nu\\
& + 8\bar\psi\gamma^{\mu\nu}\partial_\mu\chi_\nu + 8\bar\psi_\mu\gamma^{\mu\nu}\partial_\nu\chi\\
& - \frac{2}{3}M^2 - \frac{2}{3}N^2 + \frac{2}{3}P^2 + \frac{2}{3}A_\mu A^\mu\Big\}\ ,
\end{aligned}
\tag{4.2.66}
$$

where $q = \eta^{\mu\nu}q_{\mu\nu}$ and $G_{\mu\nu}(q)$ is the linearized Einstein tensor. We observe that the action is non-diagonal in the bosonic fields $(q_{\mu\nu}\,, V_\mu\,, \phi)$ and the fermionic fields $(\psi_\mu\,, \chi)$ and $(\chi_\mu\,, \psi)$.

Finally, we fix all Stückelberg symmetries by imposing the gauge conditions

$$
\phi = \text{const}\ ,\qquad V_\mu = 0\ ,\qquad \psi = 0\ ,\qquad \chi = 0\ . \tag{4.2.67}
$$

[10] The 4D analogue of this multiplet, in superfield language, can be found in [47].

Taking into account the compensating gauge transformations

$$
\begin{aligned}
\Lambda_3 &= 0 \,, \\
\Lambda_\mu &= \frac{1}{m}\bar{\epsilon}\chi_\mu \,, \\
\eta_1 &= -\frac{1}{12m}\left(M-2P\right)\epsilon \,, \\
\eta_2 &= \frac{1}{12m}\left(N+\gamma^\rho A_\rho\right)\epsilon \,,
\end{aligned}
\tag{4.2.68}
$$

we obtain the final form of the supersymmetry rules of the 3D $\mathcal{N}=1$ off-shell massive spin-2 multiplet:

$$
\begin{aligned}
\delta q_{\mu\nu} &= \bar{\epsilon}\gamma_{(\mu}\psi_{\nu)} + \frac{1}{m}\bar{\epsilon}\partial_{(\mu}\chi_{\nu)} \,, \\
\delta\psi_\mu &= -\frac{1}{4}\gamma^{\rho\lambda}\partial_\rho q_{\lambda\mu}\epsilon + \frac{1}{12}\gamma_\mu(M+P)\epsilon + \frac{1}{12m}\partial_\mu(N+\gamma^\rho A_\rho)\epsilon \,, \\
\delta\chi_\mu &= \frac{1}{4}m\gamma^\rho q_{\rho\mu}\epsilon + \frac{1}{4}A_\mu\epsilon - \frac{1}{12}\gamma_\mu(N+\gamma^\rho A_\rho)\epsilon - \frac{1}{12m}\partial_\mu(M-2P)\epsilon \,, \\
\delta M &= \bar{\epsilon}\gamma^{\rho\lambda}\partial_\rho\psi_\lambda - m\bar{\epsilon}\gamma^\rho\chi_\rho \,, \\
\delta N &= -\bar{\epsilon}\gamma^{\rho\lambda}\partial_\rho\chi_\lambda + m\bar{\epsilon}\gamma^\rho\psi_\rho \,, \\
\delta P &= \frac{1}{2}\bar{\epsilon}\gamma^{\rho\lambda}\partial_\rho\psi_\lambda + m\bar{\epsilon}\gamma^\rho\chi_\rho \,, \\
\delta A_\mu &= \frac{3}{2}\bar{\epsilon}\gamma_\mu{}^{\rho\lambda}\partial_\rho\chi_\lambda - \bar{\epsilon}\gamma_\mu\gamma^{\rho\lambda}\partial_\rho\chi_\lambda - \frac{1}{2}m\bar{\epsilon}\gamma_\mu{}^\rho\psi_\rho + m\bar{\epsilon}\psi_\mu \,.
\end{aligned}
\tag{4.2.69}
$$

These transformation rules leave the following action invariant:

$$
\begin{aligned}
S_{\text{SUSY FP}} = \int d^3x \Big\{ & q^{\mu\nu}G_{\mu\nu}(q) - m^2\left(q^{\mu\nu}q_{\mu\nu} - q^2\right) \\
& -4\bar{\psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - 4\bar{\chi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\chi_\rho + 8m\bar{\psi}_\mu\gamma^{\mu\nu}\chi_\nu \\
& -\frac{2}{3}M^2 - \frac{2}{3}N^2 + \frac{2}{3}P^2 + \frac{2}{3}A_\mu A^\mu \Big\} \,.
\end{aligned}
\tag{4.2.70}
$$

The above action (4.2.70) and its supersymmetry rules (4.2.69) are the off-shell version of the massive part of (4.2.25) and (4.2.26).

Massless limit

Now we discuss the massless limit ($m \to 0$) of the supersymmetric FP theory. Like what we did in the Proca case, in order to take this limit and avoid unwanted factors of $1/m$, we need to go back to the formulation before the Stückelberg symmetries were fixed, i.e. the action (4.2.66) and its transformation rules (4.2.65), whose massless limit is well-defined. This is equivalent to making the following field redefinitions in the final action (4.2.70) and transformation rules (4.2.69)

$$\begin{aligned} q_{\mu\nu} &= \tilde{q}_{\mu\nu} - \frac{2}{m}\partial_{(\mu}V_{\nu)} - \frac{1}{m^2}\partial_\mu\partial_\nu\phi \ , \\ \psi_\mu &= \tilde{\psi}_\mu - \frac{1}{m}\partial_\mu\psi \ , \\ \chi_\mu &= \tilde{\chi}_\mu + \frac{1}{m}\partial_\mu\chi \ , \end{aligned} \tag{4.2.71}$$

and then renaming the fields by removing the tildes.

One can check that the massless limit of the supersymmetry rules in (4.2.65) will become more manifest in its multiplet structure, if we do the following field redefinitions

$$S = \frac{1}{6}(M+P) \quad , \quad F = \frac{4}{3}(M-2P) \ . \tag{4.2.72}$$

Moreover, in order to diagonalize the massless limit of the action (4.2.66), we further need to do the following redefinitions

$$q_{\mu\nu} = q'_{\mu\nu} - \eta_{\mu\nu}\phi \ , \ \psi_\mu = \psi'_\mu + \gamma_\mu\chi \ , \ S = S' - \frac{1}{8}F \ , \ \chi_\mu = \chi'_\mu - \gamma_\mu\psi \ . \tag{4.2.73}$$

Thus the massless limit of the action is finally converted into

$$\begin{aligned} S_{\text{SUSY FP', } m\to 0} = \int d^3x \Big\{ & q'^{\mu\nu}G_{\mu\nu}(q') - 4\bar{\psi}'_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi'_\rho - 8S'^2 \\ & - F^{\mu\nu}F_{\mu\nu} - \frac{2}{3}N^2 + \frac{2}{3}A^\mu A_\mu - 4\bar{\chi}'_\mu\gamma^{\mu\nu\rho}\partial_\nu\chi'_\rho - 8\bar{\psi}\partial\!\!\!/\psi \\ & - 2\partial_\mu\phi\partial^\mu\phi - 8\bar{\chi}\partial\!\!\!/\chi + \frac{1}{8}F^2 \Big\} \ . \end{aligned} \tag{4.2.74}$$

This action can be written as the sum of three actions for the three multiplets: $(q'_{\mu\nu}, \psi'_\mu, S')$, $(V_\mu, \chi'_\mu, \psi, N, A_\mu)$ [11] and (ϕ, χ, F). By dropping some terms that can be absorbed into gauge transformations, their supersymmetry rules can be derived as the following:

$$\begin{aligned} \delta q'_{\mu\nu} &= \bar{\epsilon}\gamma_{(\mu}\psi'_{\nu)} \ , \\ \delta\psi'_\mu &= -\frac{1}{4}\gamma^{\rho\sigma}\partial_\rho q'_{\mu\sigma}\epsilon + \frac{1}{2}S'\gamma_\mu\epsilon \ , \\ \delta S' &= \frac{1}{4}\bar{\epsilon}\gamma^{\mu\nu}\partial_\mu\psi'_\nu \ , \end{aligned} \tag{4.2.75}$$

[11] An on-shell version of this multiplet was introduced in [48].

$$
\begin{aligned}
\delta V_\mu &= \bar{\epsilon}\gamma_\mu\psi - \frac{1}{2}\bar{\epsilon}\chi'_\mu \ , \\
\delta\psi &= -\frac{1}{8}\gamma^{\rho\lambda}F_{\rho\lambda}\epsilon - \frac{1}{12}N\epsilon - \frac{1}{12}\gamma^\alpha A_\alpha\epsilon \ , \\
\delta\chi'_\mu &= -\frac{1}{4}\gamma^\alpha F_{\alpha\mu}\epsilon - \frac{1}{8}\gamma_\mu\gamma^{\rho\lambda}F_{\rho\lambda}\epsilon - \frac{1}{6}\gamma_\mu N\epsilon + \frac{1}{4}A_\mu\epsilon - \frac{1}{6}\gamma_\mu\gamma^\alpha A_\alpha\epsilon \ , \\
\delta N &= \bar{\epsilon}\gamma^\alpha\partial_\alpha\psi - \bar{\epsilon}\gamma^{\alpha\beta}\partial_\alpha\chi'_\beta \ , \\
\delta A_\mu &= \frac{3}{2}\bar{\epsilon}\gamma_\mu{}^{\alpha\beta}\partial_\alpha\chi'_\beta - \bar{\epsilon}\gamma_\mu\gamma^{\alpha\beta}\partial_\alpha\chi'_\beta + \bar{\epsilon}\gamma_\mu{}^\alpha\partial_\alpha\psi + \bar{\epsilon}\partial_\mu\psi \ , \qquad (4.2.76)
\end{aligned}
$$

and

$$
\begin{aligned}
\delta\phi &= -\bar{\epsilon}\chi \ , \\
\delta\chi &= -\frac{1}{4}\gamma^\mu\epsilon\left(\partial_\mu\phi\right) - \frac{1}{16}F\epsilon \ , \\
\delta F &= -4\bar{\epsilon}\gamma^\mu\partial_\mu\chi \ . \qquad (4.2.77)
\end{aligned}
$$

The multiplet $(q'_{\mu\nu}, \psi'_\mu, S')$ is exactly the massless spin-2 multiplet (4.2.27). As a side remark, if we couple this model with a matter source (supercurrent multiplet), then in the massless limit due to the interaction between the multiplet (ϕ, χ, F) and matter, the model behaves differently from the massless model, which is the 3D analogue of the vDVZ discontinuity in 4D.[12]

4.2.2.3 Off-Shell Lower-Derivative SNMG (Linearized)

We still do not know the full theory of the off-shell lower-derivative SNMG, but at the linearized level, we can derive its action and supersymmetry rules. Combining the action for the massless multiplet (4.2.28) with the action for the massive multiplet (4.2.70), we obtain

$$
\begin{aligned}
S_{\text{two multiplets}} = \int d^3x \Big\{ & -k^{\mu\nu}G_{\mu\nu}(k) + 4\bar{\lambda}_\mu\gamma^{\mu\nu\rho}\partial_\nu\bar{\lambda}_\rho + 8S^2 \\
& + q^{\mu\nu}G_{\mu\nu}(q) - m^2\left(q^{\mu\nu}q_{\mu\nu} - q^2\right) \\
& - 4\bar{\psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - 4\bar{\chi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\chi_\rho + 8m\bar{\psi}_\mu\gamma^{\mu\nu}\chi_\nu \\
& - \frac{2}{3}M^2 - \frac{2}{3}N^2 + \frac{2}{3}P^2 + \frac{2}{3}A_\mu A^\mu \Big\} \ , \qquad (4.2.78)
\end{aligned}
$$

which is invariant under the supersymmetric rules (4.2.27) and (4.2.69). Then if we do the inverse of the field redefinitions (4.2.24), i.e.

$$
\begin{aligned}
k_{\mu\nu} &= h_{\mu\nu} - q_{\mu\nu} \ , \\
\lambda_\mu &= \rho_\mu - \psi_\mu \ , \qquad (4.2.79)
\end{aligned}
$$

[12]For details, see [49].

we obtain finally the action:

$$\begin{aligned} S_{\text{SNMG off-shell}} &= \int d^3x \Big\{ -k^{\mu\nu}G_{\mu\nu}(k) + 4\bar{\lambda}_\mu\gamma^{\mu\nu\rho}\partial_\nu\bar{\lambda}_\rho + 8S^2 \\ &\quad + q^{\mu\nu}G_{\mu\nu}(q) - m^2\left(q^{\mu\nu}q_{\mu\nu} - q^2\right) \\ &\quad - 4\bar{\psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - 4\bar{\chi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\chi_\rho + 8m\bar{\psi}_\mu\gamma^{\mu\nu}\chi_\nu \\ &\quad - \frac{2}{3}M^2 - \frac{2}{3}N^2 + \frac{2}{3}P^2 + \frac{2}{3}A_\mu A^\mu \Big\} , \end{aligned} \tag{4.2.80}$$

and its supersymmetry rules[13]

$$\begin{aligned} \delta h_{\mu\nu} &= \bar{\epsilon}\gamma_{(\mu}\rho_{\nu)} , \\ \delta\rho_\mu &= -\frac{1}{4}\gamma^{\rho\sigma}\left(\partial_\rho h_{\mu\sigma}\right)\epsilon + \frac{1}{2}S\gamma_\mu\epsilon + \frac{1}{12}\gamma_\mu(M+P)\epsilon , \\ \delta S &= \frac{1}{4}\bar{\epsilon}\gamma^{\mu\nu}\partial_\mu\rho_\nu - \frac{1}{4}\bar{\epsilon}\gamma^{\mu\nu}\partial_\mu\psi_\nu , \\ \delta q_{\mu\nu} &= \bar{\epsilon}\gamma_{(\mu}\psi_{\nu)} + \frac{1}{m}\bar{\epsilon}\partial_{(\mu}\chi_{\nu)} , \\ \delta\psi_\mu &= -\frac{1}{4}\gamma^{\rho\lambda}\partial_\rho q_{\lambda\mu}\epsilon + \frac{1}{12}\gamma_\mu(M+P)\epsilon + \frac{1}{12m}\partial_\mu(N+\gamma^\rho A_\rho)\epsilon , \\ \delta\chi_\mu &= \frac{1}{4}m\gamma^\rho h_{\rho\mu}\epsilon + \frac{1}{4}A_\mu\epsilon - \frac{1}{12}\gamma_\mu(N+\gamma^\rho A_\rho)\epsilon - \frac{1}{12m}\partial_\mu(M-2P)\epsilon , \\ \delta M &= \bar{\epsilon}\gamma^{\rho\lambda}\partial_\rho\psi_\lambda - m\bar{\epsilon}\gamma^\rho\chi_\rho , \\ \delta N &= -\bar{\epsilon}\gamma^{\rho\lambda}\partial_\rho\chi_\lambda + m\bar{\epsilon}\gamma^\rho\psi_\rho , \\ \delta P &= \frac{1}{2}\bar{\epsilon}\gamma^{\rho\lambda}\partial_\rho\psi_\lambda + m\bar{\epsilon}\gamma^\rho\chi_\rho , \\ \delta A_\mu &= \frac{3}{2}\bar{\epsilon}\gamma_\mu{}^{\rho\lambda}\partial_\rho\chi_\lambda - \bar{\epsilon}\gamma_\mu\gamma^{\rho\lambda}\partial_\rho\chi_\lambda - \frac{1}{2}m\bar{\epsilon}\gamma_\mu{}^\rho\psi_\rho + m\bar{\epsilon}\psi_\mu , \end{aligned} \tag{4.2.81}$$

which are the off-shell version of (4.2.12) and (4.2.14).

[13] A term $\frac{1}{m}\bar{\epsilon}\partial_{(\mu}\chi_{\nu)}$ in $\delta h_{\mu\nu}$ and a term $\frac{1}{12m}\partial_\mu(N+\gamma^\rho A_\rho)\epsilon$ in $\delta\rho_\mu$ have been dropped, because they can be absorbed in the gauge transformations of $h_{\mu\nu}$ and ρ_μ.

Chapter 5

Conclusions

In this thesis, we have first introduced massive gravity (3D higher-derivative gravity in particular) as a type of modified general relativity. Then in the following chapters we have shown that the linearized New/Topologically Massive Gravity models can be extended in several ways.

In Chapter 2, we have shown the extensions to bosonic higher spins in 3D. The linearized NMG can be derived from the 3D spin-2 FP theory via a "boosting up derivative" procedure, which solves the divergenceless condition and replaces the field with the linearized Einstein tensor of the graviton. We can apply the same procedure to arbitrary bosonic higher spins in 3D, which leads to higher-spin NMG-like models that are equivalent to higher-spin FP and that contain gauge symmetries. Furthermore, we have shown that in 3D under the divergenceless condition the Klein-Gordon operator can be factorized, and by using this trick we can drop one of the two helicities, which gives TMG-like models for higher spins. In this chapter, we have also explained the general methodology of constructing actions for these models, and have shown with explicit examples that auxiliary fields are often necessary for higher-spin actions. The NMG-like actions for odd spins contains ghosts, others do not. At the end of this chapter, we briefly introduced another way of "boosting up derivatives" which solves both the divergenceless and traceless conditions.

In Chapter 3, we have discussed the possibility to extend the bosonic NMG-like and TMG-like models (at the linearized level) to $D > 3$, and explained the restriction that we can only use specific representations in certain dimensions. For both the NMG-like and TMG-like models, it is crucial to avoid massless modes, otherwise ghosts would arise, and the higher-derivative models would not be equivalent to lower-derivative ones. Furthermore, for TMG-like models, it is further restricted by the requirement that the Klein-Gordon operator can be factorized. As a result, if we want to construct NMG-like and TMG-like models beyond 3D with actions, we can only use rectangular Young tableaux of height $2k-1$ in $4k-1$ dimensions, except that for spin-2 NMG-like models we can have some more options.

In Chapter 4, we have further extended in 3D the NMG-like and TMG-like models to fermions, and discussed the supersymmetric NMG model in detail. We have shown that at the linearized level the order of derivatives of the SNMG action can be lowered by introducing several auxiliary fields, and in principle this also can be done at the non-linear level, although the results are not illuminating. Furthermore, at the linearized level, we have shown how to derive 3D off-shell massive multiplets by performing Kaluza-Klein reduction from the 4D off-shell massless multiplets, and constructed the lower-derivative linearized SNMG action with off-shell supersymmetry.

There are still many interesting things to be thought about in the future. For instance, the discussions in this thesis mainly focus on the flat background, but perhaps similar models (for higher spins and/or higher dimensions) can also be constructed on dS and AdS backgrounds. With two parameters (the cosmological constant Λ and the mass m) to tune with, the resulting models must contain richer properties to explore. For another example, it would also be interesting to go beyond the free theory. NMG and TMG are models that describe gravitational force between matter sources, so it would be natural to ask, in the extended models, what kind of forces do the gauge fields exert when we couple them to other matter. We should also be motivated to enhance the abelian gauge symmetry in the extended models to non-abelian versions, in the hope that this might help us to construct non-linear models like the original NMG and TMG (in the bulk) with the full diffeomorphism symmetry[1]. Furthermore, due to various relations between massive gravity models and bi-gravity models, it might be possible to obtain some inspiration for the research of bi-gravity from the study of massive gravity, e.g. maybe we can learn something about how to supersymmetrize bi-gravity models from our knowledge of SNMG.

[1] According to [50, 51], this seems to be difficult (at least for the self-interaction of any mixed-symmetry tensor corresponding to a two-column Young tableau), but nevertheless there are still open questions, e.g. the interaction between tensors of different Young symmetries.

Appendix A

Conventions

In this appendix, we collect some conventions that might not have been clarified in the main body of the thesis.

Throughout this thesis, the abbreviations NMG and TMG are used for New Massive Gravity and Topologically Massive Gravity, respectively. We adopt the unit system that the Newton's constant G and the speed of light c are set to one, and the mostly-plus signature is always used. For convenience, the overall rescaling of an action is arbitrary. Boundary terms of actions are always dropped.

In Chapters 2, 3 and 4 (except Section 4.2.1.2), we discuss only the models that are linear and that are on a flat spacetime background. In these chapters, we use Greek letters to denote spacetime indices, and use Latin letters to denote spatial indices, except that in Section 4.2.1.2 we use Greek letters to denote curved spacetime indices, and use Latin letters to denote flat spacetime indices (Lorentz indices).

A.1 Fields in General Relativity

In this section we give our definition of some frequently used fields in general relativity. Torsion is not considered, unless otherwise specified.

A.1.1 Non-Linear Fields

The Christoffel symbol (without torsion) is

$$\Gamma^{\rho}_{\mu\nu}\left(g\right) = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \ . \tag{A.1.1}$$

Denoting ${e_\mu}^a$ as the vielbein (and ${e^\mu}_a$ as the inverse vielbein), then

$$g_{\mu\nu} = {e_\mu}^a \eta_{ab} {e_\nu}^b \ , \tag{A.1.2}$$

where $\eta = (-, +, \cdots, +)$.

The spin connection in terms of vielbeins reads

$$\omega_\mu{}^{ab} = 2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{|\sigma|b]}e_{\mu c}\partial_\nu e_\sigma{}^c \ . \tag{A.1.3}$$

The Riemann tensor is defined as

$$R_{\mu\nu}{}^\rho{}_\sigma = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\tau}\Gamma^\tau_{\nu\sigma} - \Gamma^\rho_{\nu\tau}\Gamma^\tau_{\mu\sigma} \ , \tag{A.1.4}$$

or equivalently

$$R_{\mu\nu ab} = \partial_\mu\omega_{\nu ab} - \partial_\nu\omega_{\mu ab} + \omega_{\mu ac}\omega_\nu{}^c{}_b - \omega_{\nu ac}\omega_\mu{}^c{}_b \ . \tag{A.1.5}$$

The Ricci tensor is

$$R_{\mu\nu} = g^{\rho\sigma}R_{\mu\rho\nu\sigma} \ , \tag{A.1.6}$$

and the curvature scalar is

$$R = g^{\mu\nu}R_{\mu\nu} \ . \tag{A.1.7}$$

We define the Einstein tensor as

$$G_{\mu\nu} = -2\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) \ , \tag{A.1.8}$$

where the factor -2 is adopted for convenience in 3D, which is different from ordinary convention.

Furthermore, the Lorentz-covariant derivative D acts on flat spacetime indices and spinor indices:

$$\begin{aligned} D_\mu V^a &= \partial_\mu V^a + \omega_\mu{}^a{}_b V^b \ , \\ D_\mu W_a &= \partial_\mu W_a + \omega_{\mu a}{}^b W_b \ , \\ D_\mu\psi &= \left(\partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\right)\psi \ . \end{aligned} \tag{A.1.9}$$

In Section 4.2.1.2, torsion is considered. We denote the spin-connection with torsion by

$$\hat{\omega}_\mu{}^{ab} = \omega_\mu{}^{ab} + \frac{1}{4}\left(2\bar{\psi}_\mu\gamma^{[a}\psi^{b]} + \bar{\psi}^a\gamma_\mu\psi^b\right) \ , \tag{A.1.10}$$

and we define $R_{\mu\nu ab}(\hat{\omega})$ to be the expression on the right-hand-side of (A.1.5) with ω replaced by $\hat{\omega}$. Furthermore we define $R_{\mu\nu}(\hat{\omega}) = e^{\rho b}e_\nu{}^a R_{\mu\rho ab}(\hat{\omega})$, $R(\hat{\omega}) = e^{\mu a}e^{\nu b}R_{\mu\nu ab}(\hat{\omega})$, $G_{\mu,\nu}(\hat{\omega}) = -2\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)$, and $G^{\rm tr}(\hat{\omega}) = g^{\mu\nu}G_{\mu,\nu}(\hat{\omega})$. Note that with the torsion contribution $R_{\mu\nu}(\hat{\omega})$ and $G_{\mu,\nu}(\hat{\omega})$ are not symmetric tensors. We also define the Lorentz-covariant derivative with torsion by $D_\mu(\hat{\omega})$ whose expression is given by (A.1.9) with ω replaced by $\hat{\omega}$.

A.1.2 Linearization

Note that in this appendix, we will use "lin" to label some linearized fields. However, in Chapters 2, 3 and 4 we drop this label, because we discuss only linearized models (except in Section 4.2.1.2).

We linearize the theory around the flat background. One way of doing this is to take the perturbation of the metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \tag{A.1.11}$$

where the graviton field is a symmetric tensor $h_{\mu\nu} = h_{\nu\mu}$. Another way is to take the perturbation of the vielbein

$$e_\mu{}^a = \delta_\mu^a + \frac{1}{2} h_{\mu,}{}^a \ , \tag{A.1.12}$$

where we use a non-symmetric tensor $h_{\mu,}{}^a$ (with a comma between its indices). At the linearized level, the vielbein becomes a Kronecker delta, so there is no difference between the Greek and Latin indices, and the above two h's are related by

$$h_{\mu\nu} = h_{(\mu,\nu)} + O(h^2), \tag{A.1.13}$$

Here we list some fields at the linearized level:

$$\Gamma^{\text{lin}}\,{}^{\rho}_{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma}\left(\partial_\mu h_{(\nu,\sigma)} + \partial_\nu h_{(\mu,\sigma)} - \partial_\sigma h_{(\mu,\nu)}\right) \ , \tag{A.1.14}$$

$$\omega^{\text{lin}}_{\mu\nu\rho} = \frac{1}{2}\partial_\mu h_{[\nu,\rho]} - \frac{1}{2}\partial_\nu h_{(\rho,\mu)} + \frac{1}{2}\partial_\rho h_{(\nu,\mu)} \ , \tag{A.1.15}$$

$$R^{\text{lin}}_{\mu\nu\rho\sigma} = -\frac{1}{2}\partial_\mu\partial_\rho h_{(\sigma,\nu)} + \frac{1}{2}\partial_\mu\partial_\sigma h_{(\rho,\nu)} + \frac{1}{2}\partial_\nu\partial_\rho h_{(\sigma,\mu)} - \frac{1}{2}\partial_\nu\partial_\sigma h_{(\rho,\mu)} \ , \tag{A.1.16}$$

$$R^{\text{lin}}_{\mu\nu} = -\frac{1}{2}\left[\Box h_{(\mu,\nu)} - \partial^\rho\left(\partial_\mu h_{(\nu,\rho)} + \partial_\nu h_{(\mu,\rho)}\right) + \partial_\mu\partial_\nu h\right] \ , \tag{A.1.17}$$

$$R^{\text{lin}} = -\Box h + \partial^\mu\partial^\nu h_{(\mu,\nu)} \tag{A.1.18}$$

$$G^{\text{lin}}_{\mu\nu} = \varepsilon_\mu{}^{\alpha\sigma}\varepsilon_\nu{}^{\beta\tau}\partial_\alpha\partial_\beta h_{(\sigma,\tau)} \quad \text{(in 3D)} \ . \tag{A.1.19}$$

where $h \equiv \eta^{\mu\nu}h_{(\mu,\nu)}$. The following formulas are also useful

$$e = 1 + \frac{1}{2}h + O(h^2) \ , \tag{A.1.20}$$

$$e^\mu{}_a = \delta_a^\mu - \frac{1}{2}h_{a,}{}^\mu + O(h^2) \ , \tag{A.1.21}$$

and it is useful to expand the Ricci scalar to the second order, i.e. $R = R^{\text{lin}} + R^{(2)} + O(h^3)$, where

$$\begin{aligned} R^{(2)} &= \eta^{\mu\rho}\eta^{\nu\sigma}R^{(2)}_{\mu\nu\rho\sigma} - \eta^{\mu\rho}h^{\nu,\sigma}R^{\text{lin}}_{\mu\nu\rho\sigma} \\ &= \frac{1}{4}h^{(\mu,\nu)}\Box h_{(\mu,\nu)} - \frac{1}{2}h^{(\mu,\nu)}\partial_\mu\partial^\rho h_{(\nu,\rho)} + \frac{1}{4}h\Box h + \ \text{(total derivative terms)} \ . \end{aligned} \tag{A.1.22}$$

Then the Einstein-Hilbert term in the linearized theory can be derived:

$$eR\left(e\right) \overset{\text{lin}}{=} \frac{1}{4} h^{(\mu,\nu)} G^{\text{lin}}_{\mu\nu}\left(h\right) \ . \tag{A.1.23}$$

A.1.3 Lorentz Transformation (Infinitesimal)

Denote the gauge parameter by $\lambda_{ab} = -\lambda_{ba}$. The infinitesimal Lorentz transformation acts on the vielbein, the spin connection and the spinor as follows:

$$\delta e_\mu{}^a = -\lambda^a{}_b e_\mu{}^b \ , \tag{A.1.24}$$

$$\delta \omega_\mu{}^{ab} = \partial_\mu \lambda^{ab} - \lambda^a{}_c \omega_\mu{}^{cb} - \lambda^b{}_c \omega_\mu{}^{ac} \ , \tag{A.1.25}$$

$$\delta\psi = -\frac{1}{4}\lambda^{ab}\gamma_{ab}\psi \ . \tag{A.1.26}$$

For the linearized theory, it becomes

$$\delta h_{\mu,\nu} = 2\lambda_{\mu\nu} \ , \tag{A.1.27}$$

$$\delta \omega^{\text{lin}}_{\mu\nu\rho} = \partial_\mu \lambda_{\nu\rho} \ , \tag{A.1.28}$$

$$\delta\psi = -\frac{1}{4}\lambda^{ab}\gamma_{ab}\psi \ . \tag{A.1.29}$$

and hence $h_{\mu,\nu}$ can be symmetrized by doing the gauge-fixing.

A.2 Other Conventions

When we use GL(n) or SO(n) to denote a group, we mean GL$(n,\mathbb{R})$ or SL$(n,\mathbb{R})$.

The Levi-Civita symbol in D-dimensional spacetime with the signature $(-,+,+,\cdots)$ satisfies the following formula:

$$\varepsilon_{\mu_1\cdots\mu_p\rho_1\cdots\rho_q}\varepsilon^{\nu_1\cdots\nu_p\rho_1\cdots\rho_q} = -p!q!\delta^{\nu_1\cdots\nu_p}_{\mu_1\cdots\mu_p} \ , \tag{A.2.1}$$

where $p+q=D$, and $\delta^{\nu_1\cdots\nu_p}_{\mu_1\cdots\mu_p} = \delta^{\nu_1}_{[\mu_1}\cdots\delta^{\nu_p}_{\mu_p]}$. We chose $\varepsilon_{012\cdots} = 1$ and $\varepsilon^{012\cdots} = -1$. Note that in curved spacetime ε is a tensor density. To make it a tensor, one should write $e^{-1}\varepsilon^{\mu_1\cdots\mu_D}$ or $e\varepsilon_{\mu_1\cdots\mu_D}$.

In the canonical analysis, we define the spatial Levi-Civita symbol $\varepsilon_{ijk\cdots} \equiv \varepsilon_{0ijk\cdots}$, and we denote $\nabla^2 = \partial^i\partial_i$. The summation convention applies to spatial indices. Each dot above a field stands for a temporal derivative ∂_0.

At the end of this appendix, we list some useful identities for fermion calculations in 3D:

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \ , \ \gamma_\mu\gamma^\mu = 3 \ , \ \gamma_\mu\gamma^\nu\gamma^\mu = -\gamma^\nu \ , \ \gamma_{\mu\nu} \equiv \gamma_{[\mu}\gamma_{\nu]} = \varepsilon_{\mu\nu\rho}\gamma^\rho \ ,$$
$$\varepsilon_{\mu\nu\rho} = \gamma_{\mu\nu\rho} \equiv \gamma_{[\mu}\gamma_\nu\gamma_{\rho]} = \gamma_{\mu\nu}\gamma_\rho + 2\eta_{\rho[\mu}\gamma_{\nu]} = \gamma_\mu\gamma_\nu\gamma_\rho - \eta_{\mu\nu}\gamma_\rho + \eta_{\mu\rho}\gamma_\nu - \eta_{\nu\rho}\gamma_\mu \ ,$$
$$\bar\psi\chi = \bar\chi\psi \ , \ \bar\psi\gamma_\mu\chi = -\bar\chi\gamma_\mu\psi \ , \ \bar\psi\gamma_{\mu\nu}\chi = -\bar\chi\gamma_{\mu\nu}\psi \ , \ \bar\psi\gamma_{\mu\nu}\gamma_\rho\chi = \bar\chi\gamma_\rho\gamma_{\mu\nu}\psi \ ,$$
$$\text{Fierz identity: } \chi\bar\psi = -\frac{1}{2}\left(\bar\psi\chi\right)\mathbf{1} - \frac{1}{2}\left(\bar\psi\gamma_\mu\chi\right)\gamma^\mu \ ,$$

where $\mathbf{1}$ is the identity matrix.

Appendix B

Action for FP Spin-s in D Dimensions

The action for generic bosonic spin-s FP equations in 4D was given by Singh and Hagen in [24] using traceless tensors. Here is the extension of their result to generic D dimensions:

$$
\begin{aligned}
S_{\text{FP spin-}s} = \int d^D x \Bigg\{ & \frac{1}{2}\phi^{(s)}\left(\Box - m^2\right)\phi^{(s)} + \frac{1}{2}s\left(\partial\phi^{(s)}\right)\left(\partial\phi^{(s)}\right) \\
& + c\Bigg\{\phi^{(s-2)}\left(\partial\partial\phi^{(s)}\right) - \frac{1}{2}\phi^{(s-2)}\left(\Box - a_2 m\right)\phi^{(s-2)} \\
& \quad + \frac{1}{2}b_2\left(\partial\phi^{(s-2)}\right)\left(\partial\phi^{(s-2)}\right) \\
& \quad - \sum_{q=3}^{s}\left(\prod_{k=2}^{q-1}c_k\right)\Bigg[\frac{1}{2}\phi^{(s-q)}\left(\Box - a_q m^2\right)\phi^{(s-q)} \\
& \quad - \frac{1}{2}b_q\left(\partial\phi^{(s-q)}\right)\left(\partial\phi^{(s-q)}\right) - m\phi^{(s-q)}\left(\partial\phi^{(s-q+1)}\right)\Bigg]\Bigg\}\Bigg\} ,
\end{aligned}
\tag{B.0.1}
$$

where $\phi^{(s)}$ is the traceless fundamental field with s symmetrized indices, ∂ stands for taking the divergence, and the parameters are given by

$$
\begin{aligned}
c &= \frac{s\left(s-1\right)\left(2s+D-6\right)}{2\left(2s+D-5\right)} , \\
a_q &= \frac{q\left(2s-2q+D\right)\left(2s-q+D-3\right)}{2\left(2s-2q+D-2\right)\left(2s-2q+D-1\right)} , \\
b_q &= \frac{\left(s-q\right)\left(2s-2q+D-4\right)}{2\left(2s-2q+D-1\right)} , \\
c_q &= \frac{\left(q-1\right)\left(s-q\right)\left(2s-2q+D-4\right)\left(2s-2q+D\right)\left(2s-q+D-2\right)}{4\left(2s-2q+D-3\right)\left(2s-2q+D-2\right)\left(2s-2q+D-1\right)} .
\end{aligned}
\tag{B.0.2}
$$

Appendix C

The Generalized 3D Cotton Tensor

The 3D linearized Cotton tensor of the graviton $h_{\mu\nu}$ is[1]

$$C_{\mu\nu}(h) = \varepsilon_{(\mu|}{}^{\sigma\tau}\partial_\sigma G_{\tau|\nu)}(h) \ .$$

In 3D it can be generalized to spin-s as a tensor constructed with derivatives of the order $2s-1$:

$$C_{\mu_1\cdots\mu_s}(h) = \varepsilon_{(\mu_1}{}^{\nu_1\rho_1}\cdots\varepsilon_{\mu_{s-1}|}{}^{\nu_{s-1}\rho_{s-1}}\partial_{\nu_1}\cdots\partial_{\nu_{s-1}}S_{\rho_1\cdots\rho_{s-1}|\mu_s)}(h) \ , \tag{C.0.1}$$

where the field h is a rank-s symmetric and traceful tensor. Here S is the spin-s generalization of the Schouten tensor which is defined in terms of the generalized Einstein tensor $G(h)$ and its traces as follows:

$$S_{\mu_1\cdots\mu_s}(h) = G_{\mu_1\cdots\mu_s}(h) + \sum_{t=1}^{[s/2]} c_t\eta_{(\mu_1\mu_2}\cdots\eta_{\mu_{2t-1}\mu_{2t}}G^{(\mathrm{tr})^t}_{\mu_{2t+1}\cdots\mu_s)}(h) \ , \tag{C.0.2}$$

where $G^{(\mathrm{tr})^t}_{\mu_{2t+1}\cdots\mu_s}(h) = \eta^{\mu_1\mu_2}\cdots\eta^{\mu_{2t-1}\mu_{2t}}G_{\mu_1\cdots\mu_s}(h)$ and where the coefficients c_t have been chosen such that under gauge transformation

$$\delta h_{\mu_1\mu_2\cdots\mu_s} = \eta_{(\mu_1\mu_2}\Lambda_{\mu_3\cdots\mu_s)} \ , \tag{C.0.3}$$

S transforms as

$$\delta S_{\mu_1\mu_2\cdots\mu_s}(h) = \partial_{(\mu_1}\partial_{\mu_2}\Omega_{\mu_3\cdots\mu_s)}(\Lambda) \ , \tag{C.0.4}$$

for some symmetric rank-$(s-2)$ tensor Ω that depends on Λ. This is the generalization of the linearized conformal transformation, under which the generalized Cotton tensor should be invariant.

The c_t coefficients satisfy the recursive relation

$$c_t = -\frac{(s-2t+2)(s-2t+1)}{4t(s-t)}c_{t-1} \ , \ c_0 = 1 \ , \tag{C.0.5}$$

[1] A more standard definition at the non-linear level should be $C_{\mu\nu} = \epsilon_\mu{}^{\sigma\tau}\nabla_\sigma\left(R_{\tau\nu} - \frac{1}{4}\eta_{\tau\nu}R\right)$, but for simplicity of the calculation, in our definition we take only the symmetric part of it, and it has been rescaled by an overall factor. In this appendix, the discussion is only at the linearized level, so here all the tensors are understood as linearized ones without being labelled.

which has the solution

$$c_t = -\frac{s}{4} \cdot \frac{P\left(1 - \frac{s}{2}, -1 + t\right) P\left(\frac{3}{2} - \frac{s}{2}, -1 + t\right)}{P\left(2, -1 + t\right) P\left(2 - s, -1 + t\right)} \ , \tag{C.0.6}$$

with $P(m, n)$ given by

$$P\left(m, n\right) = m\left(m + 1\right) \cdots \left(m + n - 1\right) \text{ for } n \in \mathbb{Z}^+ \ , \text{and } P\left(m, 0\right) = 1 \ . \tag{C.0.7}$$

One finds that in terms of these coefficients the parameter $\Omega_{\mu_3 \cdots \mu_s}$ is given by

$$\begin{aligned} \Omega_{\mu_3 \cdots \mu_s}\left(\Lambda\right) &= G_{\mu_3 \cdots \mu_s}\left(\Lambda\right) \\ &+ \sum_{t=1}^{[s/2]-1} c_t \frac{\left(s - 2t\right)\left(s - 2t - 1\right)}{s\left(s - 1\right)} \eta_{(\mu_3 \mu_4} \cdots \eta_{\mu_{2t+1} \mu_{2t+2}} G^{\mathrm{tr}^t}_{\mu_{2t+3} \cdots \mu_s)}\left(\Lambda\right) \ . \end{aligned} \tag{C.0.8}$$

Furthermore, because the construction here is purely based on the Einstein tensor, the generalized Cotton tensor also satisfies the gauge symmetry generalized from the linearized diffeomorphisms:

$$\delta h_{\mu_1 \mu_2 \cdots \mu_s} = \partial_{(\mu_1} \xi_{\mu_2 \mu_3 \cdots \mu_s)} \ . \tag{C.0.9}$$

One can also check that this generalized Cotton tensor is both traceless and divergenceless.

To summarize, this generalized Cotton tensor is a symmetric, traceless and divergenceless tensor of rank-s with derivatives of the order $2s - 1$, and it is invariant under two kinds of gauge transformations (C.0.4) and (C.0.9).

Appendix D

Young Tableaux and Symmetrizers

For the general linear group $\mathrm{GL}(n, \mathbb{R})$, each irrep is represented by a Young tableau, and each Young tableau corresponds to a certain type of tensor that has been projected with a certain symmetry on its indices.

For instance, a rank-3 tensor T_{abc} can be projected in several ways, corresponding to different Young tableaux:

$$T_{(abc)} \qquad \begin{array}{|c|c|c|}\hline a & b & c \\ \hline \end{array}$$

$$T_{[abc]} \qquad \begin{array}{|c|}\hline a \\ \hline b \\ \hline c \\ \hline \end{array}$$

$$\frac{1}{3}(T_{abc} + T_{cba} - T_{bac} - T_{cab}) \qquad \begin{array}{|c|c|}\hline a & c \\ \hline b & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}$$

where as usual we have the symmetric and antisymmetric projections, and furthermore we have the mixed symmetry projection that first symmetrizes indices in the same row (in this case a and c) and then antisymmetrizes indices in the same column (in this case a and b). The factor $\frac{1}{3}$ is such chosen that doing the projection twice gives the same result as doing it once.

In general, consider a generic Young tableau

$$\begin{array}{|c|c|ccc|c|}\cline{1-2}\cline{6-6} a_{11} & a_{12} & \cdots & & & a_{1p_1} \\ \cline{1-2}\cline{6-6} a_{21} & a_{22} & \cdots & & \multicolumn{2}{c}{} \\ \cline{1-2} \multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{\vdots} & \ddots & & \multicolumn{2}{c}{} \\ \cline{1-1} a_{q_1 1} & \multicolumn{5}{c}{} \\ \cline{1-1} \end{array} \tag{D.0.1}$$

with the index a_{ij} filled in the box on its ith row and jth column. Denote the length of the ith row as p_i ($p_1 \geq p_2 \geq p_3 \geq \cdots$) and the length of the jth column as q_j ($q_1 \geq q_2 \geq q_3 \geq \cdots$). By the standard definition, the corresponding projection that operates on a multi-form[1] $T_{a_{11}\cdots a_{q_1 1}, a_{12}\cdots a_{q_2 2},\cdots}$ is given by[2]

$$\mathcal{Y}_{[q_1, q_2, \cdots]} = \alpha \prod_j \mathcal{A}_j \prod_i \mathcal{S}_i \ , \tag{D.0.2}$$

where the operator $\mathcal{S}_i$ stands for the symmetrization $(a_{i1} \cdots a_{ip_i})$, and $\mathcal{A}_j$ stands for the antisymmetrization $[a_{1j} \cdots a_{q_j j}]$. α is a normalization factor, which makes

$$\mathcal{Y}^2 = \mathcal{Y} \ . \tag{D.0.3}$$

The expression of α can be explicitly given as

$$\alpha = \frac{\prod\limits_j (q_j!) \prod\limits_i (p_i!)}{\prod\limits_{i,j} \operatorname{hook}(i,j)} \ , \tag{D.0.4}$$

where $\operatorname{hook}(i,j)$ is the hook-length of the hook that turns at the box on the ith row and jth column, which is equal to $p_i + q_j - i - j + 1$.

When we say that T satisfies the symmetry property (D.0.1), we mean that

$$T_{a_{11}\cdots a_{q_1 1}, a_{12}\cdots a_{q_2 2},\cdots} = \mathcal{Y}_{[q_1, q_2, \cdots]} T_{a_{11}\cdots a_{q_1 1}, a_{12}\cdots a_{q_2 2},\cdots} \ , \tag{D.0.5}$$

i.e. doing the projection does not change the tensor. The number of independent components of such a tensor can be calculated by

$$\prod_{i,j} \frac{(n - i + j)}{\operatorname{hook}(i,j)} \ . \tag{D.0.6}$$

For the special orthogonal group $\mathrm{SO}(n, \mathbb{R})$, we also use Young Tableaux to label irreps, and correspondingly we use tensors projected in the same way. However, there is one crucial difference: for the $\mathrm{GL}(n, \mathbb{R})$ group we use traceful tensors, but for the $\mathrm{SO}(n, \mathbb{R})$ group we use traceless tensors. Consequently when we calculate the number of independent components of tensors that are irreps of $\mathrm{SO}(n, \mathbb{R})$, we must subtract the number of constraints imposed by the traceless condition.[3]

In some situations, by removing the trace of a tensor there is no degree of freedom left. For instance, a theorem says that, for the group $\mathrm{SO}(n, \mathbb{R})$, the first two columns of the Young tableaux contain at most n boxes, otherwise the corresponding traceless tensor is zero. If we define "spin" to be the number of columns of a Young tableau, this theorem explains why "spin" has no ambiguity for massive

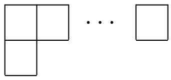
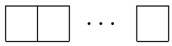
[1] A multi-from means a tensor that has multiple sets of antisymmetrized indices, which are separated by commas.

[2] The operator $\mathcal{Y}_{[q_1, q_2, \cdots]}$, which does the projection is called the Young symmetrizer. We adopt this notation from [37].

[3] For more details on Young tableaux and Young symmetrizers, including the relevant theorems mentioned in this thesis, one can read e.g. [40].

particles in 3D, because for the little group SO(2) only two boxes are allowed in the first two columns, and thus each column contains at most one box,[4] i.e. only totally symmetric tensors are allowed.

Furthermore, from this theorem one can see that only two types of Young tableaux

 and 

are allowed for massive particles in 4D, whose little group is SO(3), because at most three boxes are allowed in the first two columns. Moreover there is another theorem saying that for the group SO$(n, \mathbb{R})$ two Young tableaux are dual to each other, if they have in total n boxes in their first columns and have the same structure from the second column onwards, which means if the two types of Young tableaux shown above have the same number of columns, they are actually equivalent representations, and this explains why "spin" has also no ambiguity for massive particles in 4D.

However for massive particles in higher dimensions ambiguity arises. For instance in 5D, i.e. for the group SO(4), "spin-2" representations like ,  and  are all allowed, but they are not equivalent.

At the end of this appendix, we would like to mention another useful theorem, saying that for the GL$(n, \mathbb{R})$ group, i.e. for traceful tensors, a pair of Young tableaux that each have p_1 columns are dual to each other, if the number of boxes in the jth column of the first tableau plus the number of boxes in the $(p_1 - j + 1)$th column of the second tableau is equal to n for each value of $j = 1, 2, \cdots p_1$.

In this thesis, the generalized Einstein tensor (off-shell traceful) is always defined as the dual of the generalized Riemann tensor, and thus they have the same number of independent components. Then setting the former to zero means the latter also vanishes, and hence the gauge field is a pure gauge.

[4]One can also put two boxes only in the first column (i.e. a tensor with two antisymmetrized indices of SO(2)), but it is easy to see that this tensor is dual to a scalar, which is a trivial representation.

Appendix E

Parity and the 3D Dirac Equation

In this appendix, we discuss the parity transformation in the context of fermions. We will see that in 3D it transforms a pair of Dirac equations into each other, which is different from the 4D situation. We discuss only in flat spacetime with the mostly-plus signature.

E.1 Four-Dimensional Parity

The standard definition of the parity transformation in 4D is the reversal of all three spatial dimensions.

The transformation rule for a spinor ψ is expressed by the gamma matrix that is labelled by the unreversed dimension, i.e. γ^0:

$$\psi \to i\gamma^0\psi \ . \tag{E.1.1}$$

Because $\gamma^0\gamma^0 = -1$, an imaginary unit i is multiplied, so that performing the transformation twice will result in an identity. Furthermore, using the fact that $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, one may check that under the parity transformation

$$\bar{\psi} \to \bar{\psi}\left(i\gamma^0\right) \ , \tag{E.1.2}$$

and therefore

$$\bar{\psi}\psi \to \bar{\psi}\psi \ , \tag{E.1.3}$$

i.e. $\bar{\psi}\psi$ is a scalar.

The gamma matrices γ^μ, which carries one spacetime index and two spinor indices, should transform as

$$\gamma^\mu \to \left(i\gamma^0\right)\gamma^\nu\left(i\gamma^0\right)^{-1} P_\nu{}^\mu \ , \tag{E.1.4}$$

where

$$P_\nu{}^\mu = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \ . \tag{E.1.5}$$

One can prove that the r.h.s. of (E.1.4) is equal to γ^μ, which means γ^μ is invariant under the parity transformation. Thus the Dirac equation in 4D is invariant.

E.2 Three-Dimensional Parity

By definition the parity transformation should flip an odd number of dimensions, otherwise it is equivalent to a rotation. In 3D, there are two spatial dimensions, so it does not make sence to reverse both of them. We define the parity transformation in 3D to be the reversal of the second spatial dimension.

Now we continue the 3D discussion in a similar manner to 4D. We use the gamma matrices labelled by the unreversed dimensions to transform a spinor, i.e.

$$\psi \to \left(\gamma^0\gamma^1\right)\psi \ . \tag{E.2.1}$$

Performing the transformation twice will result in an identity. Furthermore, one may check that under the parity transformation

$$\bar{\psi} \to -\bar{\psi}\left(\gamma^0\gamma^1\right) \ , \tag{E.2.2}$$

and therefore

$$\bar{\psi}\psi \to -\bar{\psi}\psi \ , \tag{E.2.3}$$

i.e. $\bar{\psi}\psi$ is a pseudoscalar.

The gamma matrices should transform as

$$\gamma^\mu \to \left(\gamma^0\gamma^1\right)\gamma^\nu\left(\gamma^0\gamma^1\right)^{-1} P_\nu{}^\mu \ , \tag{E.2.4}$$

where

$$P_\nu{}^\mu = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} . \tag{E.2.5}$$

One can prove that the r.h.s. of (E.2.4) is equal to $-\gamma^\mu$, then we see that by a parity transformation γ^μ acquires a minus sign, which is consistent with the fact that $\varepsilon^{\mu\nu\rho} = \gamma^{\mu\nu\rho}$ acquires a minus sign. Thus the Dirac equation in 3D is not invariant. In 3D we have a pair of Dirac equations that are interchanged under the parity transformation:

$$\begin{aligned} \left(\partial\!\!\!/ - m\right)\psi &= 0 \ , \\ \left(\partial\!\!\!/ + m\right)\psi &= 0 \ . \end{aligned} \tag{E.2.6}$$

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Nederlandse Samenvatting

Deze dissertatie behandelt uitbreidingen op driedimensionale zwaartekrachtmodellen met hogere afgeleiden. Om deze modellen uit te kunnen leggen, zullen we eerst ingaan op Einsteins algemene relativiteitstheorie.

Bijna een eeuw geleden ontwikkelde Einstein de algemene relativiteitstheorie, die een revolutie teweegbracht in het denken over ruimte en tijd. De algemene relativiteitstheorie combineert ruimte en tijd in een geometrisch, vierdimensionaal object, genaamd ruimtetijd, waarvan de vorm en grootte veranderlijk zijn. De kromming van de ruimtetijd manifesteert zich als zwaartekracht. De dynamiek van de ruimtetijd wordt beschreven door Einsteins veldvergelijkingen, een stelsel van tweede orde differentiaalvergelijkingen. Omdat de voorspellingen van de algemene relativiteitstheorie met grote precisie door experimenten worden bevestigd (zo wordt bijvoorbeeld de baan van Mercurius veel preciezer beschreven door de algemene relativiteitstheorie dan door Newtons theorie), is het nu de standaard theorie voor zwaartekracht. Daarnaast gebruiken kosmologen het als de fundamentele theorie om de evolutie van het heelal te beschrijven.

Echter, de algemene relativiteitstheorie wordt ook geconfronteerd met uitdagingen. Zo blijkt uit astronomische waarnemingen dat het heelal een versnelde uitdijing ondergaat, alsof het gevuld is met een donkere energie die dingen uit elkaar drijft. De donkere energie kan in Einsteins vergelijkingen worden opgenomen middels een positieve parameter, de kosmologische constante genaamd, waarvan de waarde extreem klein is, maar niet gelijk aan nul. De algemene relativiteitstheorie werpt geen licht op de oorsprong van deze constante en op de onnatuurlijk kleine waarde. Een andere uitdaging is dat, om extreme fenomenen als zwarte gaten en het begin van het heelal te verklaren, het nodig is om de algemene relativiteitstheorie met de kwantummechanica te combineren. Dit blijkt lastig te zijn omdat, wanneer men de algemene relativiteitstheorie kwanticeert zoals gebruikelijk is in de deeltjesfysica, oneindigheden opduiken waar natuurkundigen niet mee kunnen omgaan. Om oplossingen te vinden voor deze problemen proberen natuurkundigen al decennia lang om de algemene relativiteitstheorie aan te passen.

Dit heeft een boel aangepaste modellen voor zwaartekracht in vier dimensies opgeleverd. De analyse van deze modellen kan behoorlijk ingewikkeld zijn. Daarom bestuderen theoretisch natuurkundigen soms vereenvoudigde versies van (aangepaste) algemene relativiteitstheorie als speelmodel, om een beter begrip te krijgen van de wiskundige gereedschappen die men gebruikt. Twee voorbeelden van zulke speelmodellen zijn New Massive Gravity (NMG) and Topologically Massive Gravity (TMG). Beide zijn driedimensionale hogere afgeleiden zwaartekrachtmodellen, d.w.z. aanpassingen op algemene relativiteitstheorie in drie dimensies met correctietermen die meer dan twee afgeleiden bevatten.

Deze dissertatie poogt tot een beter begrip te komen van het theoretische raamwerk van NMG en TMG door te onderzoeken of ze kunnen worden uitgebreid naar een bredere verzameling van modellen. De gelineariseerde versie van NMG en TMG (benaderende modellen voor NMG en TMG in de situatie dat de ruimtetijd vlak is en de zwaartekracht zwak is) kan uitgebreid worden in verschillende aspecten; elk hiervan wordt besproken in één hoofdstuk. Voordat we deze uitbreidingen uitleggen, moeten we het concept graviton introduceren.

In elektromagnetisme kunnen elektromagnetische golven worden opgevat als een deeltje dat foton wordt genoemd, en het foton brengt elektromagnetische krachten over. Het foton heeft geen massa (d.w.z. het reist met de constante snelheid van het licht). Hetzelfde gebeurt in zwaartekrachtmodellen. Zwaartekrachtgolven (rimpels in de ruimtetijd) kunnen worden gezien als een deeltjessoort die het graviton wordt genoemd, en die de zwaartekracht overbrengt. Het graviton van de algemene relativiteitstheorie is eveneens massaloos. Echter, in NMG en TMG is het graviton massief (d.w.z. het reist langzamer dan licht).

In de natuurkunde worden deeltjes doorgaans gerangschikt naar spin, het intrinsieke impulsmoment. Het foton bijvoorbeeld is een spin-1 deeltje, en het graviton is een spin-2-deeltje. NMG en TMG kunnen dus worden opgevat als modellen die massieve spin-2-deeltjes in een driedimensionale ruimtetijd beschrijven.

Hoofdstuk twee laat zien dat, in driedimensionale ruimtetijd, gelineariseerde NMG en TMG kunnen worden uitgebreid naar willekeurige positieve gehele spin (d.w.z. spin-1, spin-2, spin-3, etc.). Hoofdstuk drie laat zien hoe de modellen van hoofdstuk twee in bepaalde situaties verder uitgebreid kunnen worden naar sommige dimensies groter dan drie. Het eerste deel van hoofdstuk vier laat zien dat het in drie dimensies ook mogelijk is de modellen uit te breiden naar halftallige spin (spin-3/2, spin-5/2, spin-7/2, etc.). Deeltjes met heeltallige spin worden bosonen genoemd, en deeltjes met halftallige spin worden fermionen genoemd. In de theoretische natuurkunde bestaat een hypothese, supersymmetrie genaamd, die zegt dat er voor ieder type boson een type fermion als tegenhanger bestaat en vice versa. Dit idee kan toegepast worden op vele zwaartekrachtmodellen en in het tweede deel van hoofdstuk vier wordt besproken hoe NMG gesupersymmetriseerd kan worden.

Deze dissertatie eindigt met hoofdstuk vijf, waar wordt geconcludeerd dat NMG en TMG, gelineariseerd op een vlakke ruimtetijdachtergrond, inderdaad kunnen worden uitgebreid naar een veel breder raamwerk van modellen. Aan het eind van dit hoofdstuk wordt een aantal openstaande vragen voor vervolgonderzoek geopperd, bijvoorbeeld of er ook uitbreidingen mogelijk zijn op gekromde achtergronden en voor niet-lineaire modellen.

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